Inventory Adjustment and Aggregate Fluctuations

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Abstract

In this paper, we investigate the relationship between lumpy inventory adjustment and aggregate fluctuations. We present a general equilibrium model composed of many firms interacting in the input–output network. Each firm follows a threshold-type inventory policy that generates a lumpy production pattern. First, we provide a tractable method to compute the competitive equilibrium when the network structure is symmetric and all firms are homogeneous. The branching process is employed to estimate the key statistics of complex adjustment process, and we derive the stationary distribution in the long-run equilibrium. Second, we prove that the aggregate fluctuations almost surely converge to zero and the law of large numbers is satisfied in our framework. We also show that the probability distribution function of impulse response decays exponentially if the labor share is positive. We conclude that the positive labor share reduces the high inventory fluctuations and moderates the business cycle.

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1 Introduction

Business cycles have always been an important issue in economics. Innumerable studies have tried to explain how and why the economy fluctuates over time. The various theories are roughly divided into two types on the basis of whether or not microeconomic characteristics affect the aggregate economic activities. Mainstream theories emphasize the importance of macroeconomic shocks such as monetary and fiscal policy shocks, which affect all sectors simultaneously and trigger aggregate fluctuations. New theories, however, suggest the existence of channels that play an amplifying role in the propagation of micro-level shocks behind the economy-wide fluctuations.

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In the macroeconomic literature, leading theories have paid less attention to the role of micro-level shocks. Lucas (1977) argues that independent micro-level shocks would average out as the number of units approaches infinity. Lucas's argument, which supports the mainstream view, is a direct application of *the law of large numbers* to economics. According to this view, macroeconomic shocks are more important than micro shocks for the aggregate fluctuations. On the other hand, recent researchers tend not to think that macro shocks can offer a full explanation of all the fluctuations. The new view, which has been attracting much attention in the recent years, emphasizes the effects of interactions between micro units, especially input–output linkages. Indeed, in the economy composed of many independent units, a shock affecting a single unit might have little impact on the aggregate level. However, micro shocks to individual units are unlikely to be independent. If the micro units are linked, a positive shock to one unit produces a positive response among other units through this linkage. Then, small shocks are amplified and propagated in the economy through linkages to cause significant aggregate fluctuations. In this situation, the sum of these small shocks may fail to produce the kind of cancellation required by Lucas's argument.

Bak et al. (1993) is one of earlier studies following the new view, where the inventory adjustment has been considered to play an important role behind the business cycle. They focus on the fact that non-convex adjustment costs make inventory adjustment lumpy at the micro-level. They employ the threshold inventory policy through the so-called "(S, s) rule" and assume that the firms connect through a hierarchical network with a certain amount of inventories. The lumpy inventory adjustment generates micro-level perturbation, and the existence of production linkages provides a possible amplification mechanism.

Bak et al. (1993) describe the large aggregate fluctuation as critical phenomena; we refer to this line of approaches as *criticality hypothesis*. Critical phenomena are emergent properties associated with critical points at which the correlation length diverges to infinity. At the critical point, aggregate quantities follow power laws, whose properties include scale invariance and the lack of a well-defined average value. Then, micro-level shocks fail to cancel out at the aggregate level, and hence, large economic fluctuations can occur even without macro-level shocks. Their model displays self-organized criticality, and the effects of small perturbations from independent inventory adjustment fail to cancel out. The large fluctuations in aggregate productions follow a power law distribution. Nirei (2006) proposes a more general model along this line, where many individuals follow (S, s) rules and aggregate fluctuations arise from interaction with a positive feedback.

The importance of multi-sectoral linkage has also been explored in the real business cycle literature, both theoretically and empirically, after Long and Plosser (1983). More recently, Acemoglu et al. (2012) propose the *network hypothesis*, which is receiving considerable attention. They interpret the input–output structure as a weighted network, where the nodes correspond to the industries and the links to the input–output trade flows. They focus on the inequality across firms or sectors in terms of importance in an input–output network. They also derive conditions about the structure of networks to deliver low convergence rates. One of the necessary conditions is that the number of customer links follows a power law distribution with a tail exponent lower than 2. They also provide some empirical evidence that idiosyncratic shocks to the top 100 firms explain a large fraction of aggregate volatility in the United States.

Recent literature has also pointed out that the heterogeneity of firm sizes is important to understand the business cycle. Gabaix (2011) propose the *granular hypothesis* of economic fluctuations based on the fact that the distribution of firm sizes typically follows the power law. When large firms have a large share of the economy, aggregate fluctuations may arise from idiosyncratic shocks to these firms. The power law distribution makes the law of large numbers break down, and idiosyncratic shocks to large firms affect aggregate outputs.

All of these hypotheses relate to power law. Hence, in a broad sense, the large fluctuations arise because of critical phenomena in all cases. However, specifically, we interpret the criticality hypothesis to be mutually exclusive with the other two hypotheses. The criticality hypothesis claims that the power law distributions of aggregate quantities can be generated only by the system criticality, without any network or granular effect. From this viewpoint, Bak et al. (1993) provide an interesting perspective on inventory adjustment process as the source of large aggregate fluctuations. The essential point of their model is that the system criticality solely arises from the combination of lumpy inventory adjustment and local interaction through input–output linkage. In a multi-sector economy with linkage, this implies that the lumpiness attributed to non-convexity can be the source of large aggregate fluctuations by itself, even in the absence of network asymmetry or granularity of firms.

Although Bak et al.'s (1993) argument is very attractive, their results rely on the specific structure of network and production technologies. The conditions for the emergence of criticality also seem rather unique. In addition to that, the model generates wider fluctuations than those observed in reality.

The purpose of this paper is twofold. Firstly, we propose a multi-sector general equilibrium model with a non-convex cost environment as an extension of Bak et al. (1993). Then, we provide a tractable method to compute the competitive equilibrium. Along the lines of Acemoglu et al. (2012), we consider a static version of Long and Plosser (1983) but modify the model for it to contain (S, s) inventory policies that generate lumpy inventory adjustments. Owing to the complexity of the propagation mechanism through linkage, it is hard to follow the details of the micro-level lumpy adjustment process in a large economy. Instead, we employ the stochastic approach to estimate the macro-level statistics of the inventory adjustment process. Secondly, we assess the relevance of the criticality hypothesis for the study of inventory fluctuations in comparison with the other two hypotheses. We verify whether the law of large numbers applies to the model with non-convex costs, and we examine the behavior of the response to the uncorrelated micro-level shocks and how quickly their impact on aggregate volatility decays in the equilibrium. For the purpose of comparison, we get rid of the asymmetry of network structure and the heterogeneity across firms. The non-convex cost environment is the only difference between our model and the ordinary multi-sector model.

The rest of the paper is organized as follows. The next section presents a structural model with inventory adjustment and explicit production linkages. We define the competitive equilibrium and derive the aggregate relation. In section 3, we construct the representation of inventory adjustment process and provide a tractable method to estimate the key statistics. In section 4, we prove the main theorems of this study. Section 5 concludes the paper.

2 Model

2.1 Environment

2.1.1 Firms

There are a large number of firms producing differentiated goods indexed by $j \in \{1, 2, \cdots, n\}$. The produced goods of any given firm are not only purchased by consumers but also used by other firms as inputs (intermediate goods) for their own production. Each firm produces its final goods through labor and the intermediate goods purchased from other firms. The technology that produces the final goods x_j is given by the Cobb–Douglas function:

$$x_j = l_j^{(1-\gamma_j)} \prod_{i=1}^{n} x_{ij}^{a_{ij}} , \tag{1}$$

where l_j and x_{ij} denote the amount of labor hired by firm j and intermediate goods i used in the production of good j. Constant returns to scale is assumed to hold and is simply expressed by

$$\gamma_j = \sum_{i=1}^{n} a_{ij} \tag{2}$$

for all $j \in \{1, 2, \cdots, n\}$.

We assume perfectly competitive markets and price-taking firms. The firm's has two problems: a cost minimization problem and a profit maximization problem. At first, we consider a firm that chooses to minimize cost subject to producing x_j. The optimal input use problem is given by

$$c(x_j) = \min_{\{l_j, x_{ij}\}} \left\{ wl_j + \sum_{j=1}^{n} p_i x_{ij} \right\} ,$$

subject to (1) and (2). From first order conditions, cost shares of labor input and intermediate inputs are obtained, respectively, as

$$wl_j = (1 - \gamma_j)(p_j x_j) , \tag{3}$$
$$p_i x_{ij} = a_{ij}(p_j x_j) . \tag{4}$$

The assumption of constant returns to scale leads us to the linear cost functions,

$$c(x_j) = p_j x_j \tag{5}$$

for all j, where p_j is (constant) marginal cost to produce x_j. If there are no quantity discounts and the costs of holding inventories, then each firm has no incentive to store the inventory and the production takes place at the same time that the order is received.

2.1.2 Inventory Policy

Now, we introduce a substantial fixed cost to conduct production. For example, it could be the cost to activate the manufacturing device, or the cost for material transportation, and so on. In this case, the cost function is non-convex, that is,

$$\psi(x_j) = \begin{cases} \psi_j + p_j x_j & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases}, \tag{6}$$

where ψ_j is the fixed cost to produce x_j units of goods. Notice that all costs are incurred only when x_j is positive.

This type of non-convex cost function leads to the so-called "(S, s) policy" of inventory behavior. The striking features of the (S, s) policy are the fixed lot size and the lumpy pattern of productions. In the (S, s) policy, the firm constantly monitors inventories and chooses a lower level, s, below which it does not let inventories fall, and an upper level, S. Inventories are allowed to vary between two target levels, S and s. When inventory stocks reach s, the firm immediately produces to increase its inventory stocks to S. The quantity $(S - s)$ is referred to as the optimal lot size, allowing the firms to separate the timing of orders and productions. If the fixed cost is relatively large, the firm produces infrequently and the optimal lot size would be large. It is well known that the (S, s) inventory policy is effective in variety situations where non-convex costs exist. Arrow, Harris, and Marschak (1951) introduce the first mathematical formulation of the (S, s) policy, and Scarf (1960) provide a general proof of its optimality.

We suppose that each firm follows its (S_j, s_j) policy and has its own lot size $(S_j - s_j)$ associated with (6). The profit maximization generates a state-dependent behavior based on the inventory level at the beginning of a period:

$$Q_j = \begin{cases} S_j - s_j > 1 & \text{if } h_j \leq s_j \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where h_j denotes the inventory level for firm j.

We refer to the firm as *unstable* if its inventory stocks fall to its lower level or below, that is, $h_j \leq s_j$. Similarly, we refer to the firm as *stable* if $s_j < h_j \leq S_j$. The non-negative vector $\boldsymbol{h} = (h_1, \cdots, h_n)$ represents a configuration of inventory stock levels. A stable configuration is the one where all firms are stable. The set of all stable configurations are denoted as below.

$$\mathcal{S} = \left\{ \ (h_1, \cdots, h_n) \in \mathbb{Z}^n_+ \mid s_j < h_j \leq S_j, \forall j \in \{1, \cdots, n\} \ \right\}.$$

Because all (S_j, s_j) are time-invariant, the stable set $\mathcal{S}$ is also time-invariant.

The total output of firm j takes a discrete value,

$$x_j = m_j Q_j, \quad m_j \in \mathbb{N} = \{0, 1, 2, \cdots\} \tag{8}$$

where m_j is the number of productions conducted by j. The non-negative integer vector $\boldsymbol{m} = (m_1, \cdots, m_n) \in \mathbb{Z}^n_{\geq 0}$ summarizes the number of productions.

Let l^*_j denote the amount of labor used to produce the optimal lot size $(S_j - s_j)$. The labor income from one lot size is given by $wl^*_j = (1-\gamma_j)[p_j(S_j - s_j) - \psi_j]$. Henceforth, we suppose that the fixed cost ψ_j is relatively small enough to be ignored at the aggregate level. Then, let us consider

$$Y \equiv w \sum_{j=1}^{n} l^*_j m_j = \sum_{j=1}^{n} (1 - \gamma_j) p_j (S_j - s_j) m_j \tag{9}$$

as an aggregate income and aggregate output.

In a sufficiently large economy, the aggregate income Y can be regarded as a stochastic variable. Each firm produces a large amount of goods at a time so that the production pattern becomes lumpy and fluctuates more than external demands. This lumpiness and the fluctuations propagate through the network of intermediate inputs. Any firm needs to produce some units of output from other firms. Once a production takes place, it must reduce the inventory levels of other goods. They may generate inventory stock-out of other firms and trigger other productions. The chain of productions may occur and continue until all firms' inventory levels are recovered. The total number of productions is sensitively dependent on the state of initial inventory configuration. It is possible to create a big chain of productions when there exists a big cluster of low-level inventory stocks in the initial state.

2.1.3 Consumer

The representative consumer has Cobb–Douglas preferences over n distinct goods,

$$u(c_1, c_2, \cdots, c_n) = \Theta_n \prod_{i=1}^{n} c_i^{\theta_i}\,, \tag{10}$$

where c_i is the consumption of good i and Θ_n is a normalization constant discussed below. The preference weights $\{\theta_i\}$ become known before making a decision, and are given by

$$\theta_i = \frac{1}{n}\exp(\varepsilon_i)\,, \tag{11}$$

where $\{\varepsilon_i\}$ are bounded by $-\infty < \underline{\varepsilon} \leq \varepsilon_i \leq \overline{\varepsilon} < \infty$. We suppose $\sum_i \exp(\varepsilon_i) = n$.

The representative consumer is endowed with one unit of labor, which is supplied inelastically. She supplies $1/n$ units of labor to the firm j and receives labor income wm_j/n from firm j in proportion to the number of its productions. The total labor income is $Y = (w/n)\sum_{j=1}^{n} m_j$. The representative consumer is also considered to be a price taker, and maximizes her utility (10) by choosing a consumption bundle $(c_1, c_2, \cdots, c_n)$ subject to her budget constraint,

$$\sum_{i=1}^{n} p_i c_i = E[Y] \tag{12}$$

where $E[\,Y\,]$ is the expectation of aggregate income. The first-order necessary conditions imply that consumer expenditures on individual goods are proportional to their respective preference weights:

$$\frac{p_i c_i}{\theta_i} = \frac{p_j c_j}{\theta_j}, \;\; \forall i, j \in \{1, 2, \cdots, n\}\,.$$

Substituting this into the consumer budget constraint (12), we have demand functions for goods,

$$p_i c_i = \theta_i E[Y] \tag{13}$$

for all $i \in \{1, 2, \cdots, n\}$.

The indirect utility function is obtained by substituting demand functions into the utility function:

$$U = \Theta_n \prod_{i=1}^{n} \left(\frac{\theta_i E[Y]}{p_i} \right)^{\theta_i} = \Theta_n E[Y] \prod_{i=1}^{n} \left(\frac{\theta_i}{p_i} \right)^{\theta_i} .$$

By setting the utility shift parameter as $\Theta_n \equiv \prod_{i=1}^{n} (\theta_i)^{-\theta_i}$, the ideal price index is given by

$$P \equiv \prod_{i=1}^{n} p_i^{\theta_i} . \tag{14}$$

Then, it turns out that real planned aggregate expenditure,

$$y^* \equiv E[Y]/P \tag{15}$$

provides the welfare criteria and we refer to this as an *effective demand*.

2.2 Network and Assumptions

The structure of the input–output network can be represented by a weighted, directed graph $\mathcal{G} = (V, E, A)$. In the graph $\mathcal{G}$, each vertex, v_j, corresponds to the firm j in this economy so that the vertex set is $V = \{1, 2, \cdots, n\}$. A directed edge (or arc) $(i, j) \in E$ is an ordered pair of vertices, and the weight $a_{ij} \geq 0$ assigned to its directed edge represents an input flow from vertex i to vertex j. The non-negative matrix $A_n \in \mathbb{R}^{n \times n}_{\geq 0}$ with entry a_{ij} is a weighted adjacency matrix of the graph $\mathcal{G}$. As we shall see later, under Cobb–Douglas technologies and competitive factor markets, a_{ij} also corresponds to the entries of input–output tables, measuring the value of spending on input i per value of produced good j. The entry takes a positive value, $a_{ij} > 0$, if the share of firm i's output in firm j's production is positive, and $a_{ij} = 0$ otherwise. By definition of the model, self-loops $(j, j) \in E$ could be allowed whenever $a_{jj} > 0$.

We also introduce the notions of in-degree and out-degree to weighted digraphs. In a weighted digraph $\mathcal{G}$, the in-degree of v_j refers to the number of weighted arcs incident to (i.e., directed towards) v_j. Similarly, the out-degree of v_j refers to the number of weighted arcs incident from (i.e., directed away from) v_j. The cost-shares γ_j for firm j coincide with its in-degree.

We have already made suitable specializations of labor supply and production technologies. These assumptions are widely acceptable and make our model exactly similar to that of Acemoglu et al. (2012). Now, we impose other assumptions on the network structure. In order to verify the criticality hypothesis to hold without *network effect* or *granular effect*, we get rid of the asymmetry of the network structure and the heterogeneity of firms.

Assumption 2.1. *We assume the following:*

(i) *The cost shares of labor are equal for all firms:* $\gamma_j = \gamma$.

(ii) *All input sizes are equal to one if it is used in production:*

$$x_{ij} = \begin{cases} 1 & \textit{if } (i, j) \in E, \\ 0 & \textit{otherwise.} \end{cases}$$

(iii) *The optimal inventory policies are homogeneous, and identical lot sizes take positive integer value:* $(S - s) \in \mathbb{Z}_{\geq 0}$.

Assumptions 2.1 (i) (ii) are also imposed by Acemoglu et al. (2012). Assumption 2.1 (i), which means in-degree equivalence, can be regard as a condition for uniform rate of profit. This would be natural when the economy is in the long-run equilibrium. Assumption 2.1 (ii) is additionally imposed to make the model more tractable. In addition to these assumptions, we make 2.1 (iii), that is, the homogeneity of inventory policies. Assumption 2.1 states that firms are homogeneous in all ways except through the input–output structure. The structure of the input–output network has no inequality without network formation between firms.

Under Assumption 2.1, each price and the price index are given by $p = w/(1-\gamma)(S-s)n$ and $P \equiv \prod_{i=1}^{n} p_i^{\theta_i} = p$. The real total income and the real total output are expressed by

$$y_n \equiv \frac{Y_n}{P} = \frac{w}{np} \sum_{j=1}^{n} m_j = (1-\gamma)(S-s) \sum_{j=1}^{n} m_j \,, \tag{16}$$

which we refer to as an *actual output*.

2.3 Competitive Equilibrium

In this section, we define the competitive equilibrium with inventory and derive the equilibrium quantities under Assumption 2.1. The competitive equilibrium is defined as follows.

Definition 2.1. *A competitive equilibrium consists of prices* $(p_1, p_2, \cdots, p_n)$, *wage* w, *consumption bundle* $(c_1, c_2, \cdots, c_n)$ *and quantities* $(l_j, x_j, (x_{ij}))$ *such that*

(i) $\{c_i\}_{i=1}^{n}$ *solves the utility maximization problem of a representative consumer, taking* $\{p_i\}_{i=1}^{n}$ *and* w *as given.*

(ii) $\{x_{ij}\}_{i=1}^{n}$ *and* l_j *solve the profit maximization problem of firm* j *for all* $j \in \{1, 2, \cdots, n\}$; *taking* $\{p_j\}_{j=1}^{n}$ *and* w *as given.*

(iii) *labor and commodity markets are clear, that is,*

$$\sum_{i=1}^{n} l_i = 1 \,, \tag{17}$$

$$x_i = \sum_{j=1}^{n} x_{ij} + c_i + (h_i - \bar{h}_i), \quad (i = 1, 2, \cdots, n) \tag{18}$$

where $\bar{h}_i$ *denotes initial inventories for firm* i.

(iv) *All firms are stable in terms of inventory level, that is,*

$$\boldsymbol{h} = (h_1, h_2, \cdots, h_n) \in \mathcal{S} \,. \tag{19}$$

Notice that Definition 2.1 (iv) implies the terminal condition for inventory adjustment process. If there are any firms out of inventory, then the productions must take place to recover their stock levels according to the (S, s) policies. Those productions vary inventory configuration and may cause subsequent stock-outs, so that the existence of stock-outs is inconsistent with equilibrium. This additional condition for competitive equilibrium gives a distinction between the model and the ordinary multi-sector model.

We express the general equilibrium by a closed system of equations. Multiplying both sides of the market clearing condition (18) for good i by its price p_i, and plugging in demand function (13) and factor demand function (4) for good i, we obtain

$$p_i x_i = \sum_{j=1}^{n} a_{ij}(p_j x_j) + \theta_i E[Y] + p_i(h_i - \bar{h}_i)\,.$$

Let $x_j = (S_j - s_j)m_j$ denote the amount of products for j, and we have the following closed system of equations,

$$\begin{bmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} p_1(S_1-s_1)m_1 \\ p_2(S_2-s_2)m_2 \\ \vdots \\ p_n(S_n-s_n)m_n \end{bmatrix} = E[Y] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} + \begin{bmatrix} p_1(h_1-\bar{h}_1) \\ p_1(h_2-\bar{h}_2) \\ \vdots \\ p_1(h_n-\bar{h}_n) \end{bmatrix}.$$

Hereafter, we suppose the assumption 2.1. Summing up the rows of the closed system and dividing both sides by the price index $P = p$, we obtain the aggregate relation in the competitive equilibrium as follows:

$$y_n = y^* + \sum_{i=1}^{n}(h_i - \bar{h}_i). \tag{20}$$

Equation (20) shows the *ex-post* identity. The left-hand side is the actual output given by (16) and the first term of the right-hand side denotes the effective demands given by (15). IF the second term, the unplanned inventory investment, is zero, then the actual output is equal to aggregate demands. Therefore, the aggregate fluctuations in this model have the same meaning as the unplanned inventory fluctuations.

We consider an economy in which there are a large number of homogeneous firms that face a given demand distribution. All firms select the same S and s points. However, during any particular period, each firm inherits a different initial inventory stock, $\bar{h}_j$, and receives a different order, not only as final goods c_j but also as intermediate goods x_{ij}. The distribution of the initial inventory holdings results in a little difference in the micro-economic state.

In order to solve equilibrium quantities, we need to know the vector $\boldsymbol{m} = (m_1, m_2, \cdots, m_n)$ as a function of underlying parameters $A_n = [a_{ij}]$ and $\bar{\boldsymbol{h}} = (h_1, \cdots, h_n)$, when the demand shares take a certain vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$. If it were the linear model, we could obtain the solution by inverse matrix $[\,\mathbb{I} - A\,]^{-1} = \mathbb{I} + A + A^2 + \cdots$, which is called the matrix multiplier. However, this method, owing to the complexity of lumpiness of adjustment, cannot be applied to the model where the adjustment process is governed by (S, s)-type inventory policies.

3 Inventory Adjustment Process

3.1 Operator and Dynamics

In counting $\boldsymbol{m}$, we need to pay attention to the action of individual productions. Whether or not a firm produces goods depends on the level of inventory. Recall that the firms are divided into two types on the basis of the inventory: *stable* firms and *unstable* firms. Each production takes place only by an unstable firm. A stable firm never produce goods at that moment. All firms monitor their own inventory levels to avoid stock-outs at each period and want to maintain the stability of their inventory level. When a firm becomes unstable, the production takes place immediately to restore to its target level. In producing goods, the firm needs certain units of goods from other firms. A single production conducted by firm j increases its inventory level by $(S-s)$, while it decreases the inventory level at all associated firms by $(S-s)a_{ij}$. This inter-dependency of inputs is captured by the matrix $[\mathbb{I}_n - A_n]$.

Here, we define the production operator ϕ_j, which describes the action of inventory adjustment caused by single production of firm j.

Definition 3.1. *The production operator is a mapping $\phi_j : \mathbb{Z}^n_+ \to \mathbb{Z}^n_+$, that is,*

$$\phi_j(\boldsymbol{h}) = \begin{cases} \boldsymbol{h} + (S-s)\,[\,\mathbb{I}_n - A_n\,]\,\boldsymbol{e}_j & (h_{jt} < s_j) \\ \boldsymbol{h} & (s_j \leq h_{jt}) \end{cases} \tag{21}$$

where $\boldsymbol{e}_j = (0, \cdots, 1, \cdots, 0)$ is the unit vector in the direction of the j-axis in a Cartesian coordinate system.

Although the operator ϕ_j is expressed in terms of the Leontief matrix $[\mathbb{I}_n - A_n]$, ϕ_j is not linear itself. The operator ϕ_j consists of two cases. If the firm j is unstable, then mapping ϕ_j means the addition of the j-th column of $(S-s)[\mathbb{I}_n - A_n]$ to the given inventory configuration $\boldsymbol{h}$. If the firm j is stable, then ϕ_j is ineffective. In the former case, we say that the operator ϕ_j is *effective*, and in the latter case, we say that ϕ_j is *ineffective*.

Using these operators, the dynamics of inventory adjustment process can be represented by

$$\boldsymbol{h}_t = \prod_{j=1}^{n} \phi_j(\boldsymbol{h}_{t-1}) = \boldsymbol{h}_{t-1} + (S-s)[\,\mathbb{I}_n - A_n\,]\,\boldsymbol{m}_t\ , \tag{22}$$

where $\boldsymbol{m}_t$ denotes the vector of m_j obtained by composite applications of $\phi_1\phi_2\cdots\phi_n$ in t. If the corresponding operator ϕ_j is effective, then $m_j = 1$; otherwise, $m_j = 0$.

The next example shows us how the adjustment process is carried out and how $\boldsymbol{m}$ is constructed by (22).

Example 3.1. *Suppose that $(S, s) = (3, 0)$ and the adjacency matrix*

$$A_6 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The optimal lot-size is given by $(S-s)=3$*. Since in-degree is* $\gamma=\sum_{i=1}^{6}a_{ij}=2/3$ *, the number of input nodes is given by* $\gamma(S-s)=2$*. Further, suppose that the initial inventory configuration is given by* $\boldsymbol{h}_0=\{1,1,0,1,1,2\}$*. For simplicity, let us focus on effective operations. Then, the time evolution of inventory configuration is described as follows:*

$$\begin{array}{rrlcl}
 & \boldsymbol{h}_0 = & {}^t(1,1,0,1,1,2) & & \\
\boldsymbol{h}_1 = & \phi_3\boldsymbol{h}_0 = & {}^t(1,0,3,0,1,2) & \Leftrightarrow & \boldsymbol{m}_1 = {}^t(0,0,1,0,0,0)\neq\boldsymbol{0} \\
\boldsymbol{h}_2 = & \phi_2\phi_4\boldsymbol{h}_1 = & {}^t(0,3,1,3,0,2) & \Leftrightarrow & \boldsymbol{m}_2 = {}^t(0,1,0,1,0,0)\neq\boldsymbol{0} \\
\boldsymbol{h}_3 = & \phi_1\phi_5\boldsymbol{h}_2 = & {}^t(3,2,1,2,3,0) & \Leftrightarrow & \boldsymbol{m}_3 = {}^t(1,0,0,0,1,0)\neq\boldsymbol{0} \\
\boldsymbol{h}_4 = & \phi_6\boldsymbol{h}_3 = & {}^t(2,2,1,2,2,3) & \Leftrightarrow & \boldsymbol{m}_4 = {}^t(0,0,0,0,0,1)\neq\boldsymbol{0} \\
\boldsymbol{h}_5 = & \boldsymbol{h}_4 = & {}^t(2,2,1,2,2,3) & \Leftrightarrow & \boldsymbol{m}_5 = {}^t(0,0,0,0,0,0)=\boldsymbol{0} \\
 & & \vdots & & \vdots
\end{array}$$

In the beginning of the process, firm 3 is unstable so that only ϕ_3 *is the effective operator to* $\boldsymbol{h}_0$*. At* $t=1$*,* $\phi_3\boldsymbol{h}_0$ *yields another unstable configuration* $\boldsymbol{h}_1$ *with two unstable firms in position 2 and 4. At* $t=2$*, both* ϕ_2 *and* ϕ_4 *are effective, but* $\boldsymbol{h}_2=\phi_2\phi_4\boldsymbol{h}_1$ *is still unstable. At* $t=3$*, both* ϕ_1 *and* ϕ_5 *are effective, but* $\boldsymbol{h}_3=\phi_1\phi_5\boldsymbol{h}_1$ *is still unstable. At* $t=4$*, effective application of* ϕ_6 *gives stable configuration* $\boldsymbol{h}_4=\phi_6\boldsymbol{h}_3$*. For* $t\geq 5$*, there is no effective operation, which means that inventory configurations are invariant* $\boldsymbol{h}_4=\boldsymbol{h}_5=\cdots=\boldsymbol{h}_\infty$ *and* $\boldsymbol{m}_5=\cdots=\boldsymbol{m}_\infty=\boldsymbol{0}$*.*

3.2 Branching Process

In Example 3.1, $\{\boldsymbol{m}_t\}_{t=0}^{\infty}$ can be solved by hand because the number of firms, n, is very small. In general, however, it is hard to get the exact solution of $\{\boldsymbol{m}_t\}_{t=0}^{\infty}$ when n is very large. Therefore, we substitute to estimate the aggregation of vector $\boldsymbol{m}_t$ to count its exact components.

In particular, we consider the case where the adjustment process starts with one unstable firm. Now, let

$$M_t \equiv \sum_{j=1}^{n} m_{jt}\,. \tag{23}$$

denote the sum of all components of vector $\boldsymbol{m}_t$. The sequence $\{1, M_1, M_2, \cdots\}$ represents the aggregate quantities of inventory adjustment process. A simple way to account for the statistics of M_t is through the branching process. The branching process is constructed in the following manner. Let ξ denote the number of unstable firms generated by effective operation of ϕ. From Assumption 2.1, each firm has the same in-degree, $\sum_i x_{ij}=\gamma(S-s)$, and then, ξ is identical for all ϕ. By use of ξ, we can construct the sequence $\{M_t\}_{t=1}^{\infty}$ as follows. At $t=0$, by definition, the number of unstable firms is one: $M_0=1$. Thus, single production takes place at $t=1$ and generates ξ unstable firms. Then, $M_1=\xi_1$. M_1 times of production take place at $t=2$, and each single production generates ξ_j unstable firms. Then, $M_2=\xi_1+\cdots+\xi_{M_1}$. Similarly, $M_3=\xi_1+\cdots+\xi_{M_2}$ at $t=3$. In the same manner, the sequence $\{M_t\}_{t=1}^{\infty}$ can be obtained from the recursive equation:

$$M_0 = 1 \quad \text{and} \quad M_t = \sum_{j=1}^{M_{t-1}} \xi_j \; . \tag{24}$$

The next example describes the method to construct the sequence $\{M_t\}_{t=1}^{\infty}$.

Example 3.2. *Consider the same situation as in Example 3.1. Since the number of input nodes is 2, then $\xi \in \{0,1,2\}$. The branching process representation of the inventory adjustment process is obtained as follows:*

$$\begin{array}{rrlcllr}
 & \boldsymbol{h}_0 = & {}^t(1,1,0,1,1,2) & \Leftrightarrow & M_0 & & =1 \\
\boldsymbol{h}_1 = & \phi_3\boldsymbol{h}_0 = & {}^t(1,0,3,0,1,2) & \Leftrightarrow & M_1 = \xi_1 & & =2 \\
\boldsymbol{h}_2 = & \phi_2\phi_4\boldsymbol{h}_1 = & {}^t(0,3,1,3,0,2) & \Leftrightarrow & M_2 = \xi_1 + \xi_2 & = 1+1 & =2 \\
\boldsymbol{h}_3 = & \phi_1\phi_5\boldsymbol{h}_2 = & {}^t(3,2,1,2,3,0) & \Leftrightarrow & M_3 = \xi_1 + \xi_2 & = 0+1 & =1 \\
\boldsymbol{h}_4 = & \phi_6\boldsymbol{h}_3 = & {}^t(2,2,1,2,2,3) & \Leftrightarrow & M_4 = \xi_1 & & =0 \\
\boldsymbol{h}_5 = & \boldsymbol{h}_4 = & {}^t(2,2,1,2,2,3) & \Leftrightarrow & M_5 & & =0 \\
 & & \vdots & & \vdots & &
\end{array}$$

Whether or not a firm would become unstable entirely depends on the respective firms' inventory levels. In addition to the division of firms into stable and unstable ones, we divide the stable firms into two types: *critical* and *not critical* firms. We refer to the stable firm as *critical* if its inventory is just at the critical level, that is, $s_j = h_j$. Let π be the ratio of critical firms,

$$\pi \equiv \frac{\#\text{ of critical firms}}{n} \; .$$

The probability that a certain firm becomes unstable twice in the same adjustment process is of the order of $1/n$. If n is sufficiently large, we can eliminate the possibility that the chain of productions forms loops.

In the large economy, ξ can be regarded as a random variable according to π. The characteristics of the stochastic process $\{M_t\}_{t=1}^{\infty}$ depend on the probability distribution of ξ. Under Assumption 2.1, the number of firms who supply to firm j is equal to its in-degree, $\sum_{i=1}^{n} x_{ij} = \gamma(S-s)$. Since all firms are homogeneous and ξ are i.i.d., then the probability distribution of ξ is given by the binomial distribution,

$$p_k = P(\xi = k) = \begin{pmatrix} \gamma(S-s) \\ k \end{pmatrix} \pi^k (1-\pi)^{\gamma(S-s)-k}, \tag{25}$$

where $k \in \{0, 1, \cdots, \gamma(S-s)\}$ denotes the number of unstable firms generated by a single production. Using the binomial theorem, the probability generating function of ξ is given by

$$f(z) = \sum_{k=0}^{\infty} p_k z^k = (\pi z + (1-\pi))^{\gamma(S-s)} \; . \tag{26}$$

The mean of ξ is finite and given by $E[\,\xi\,] = f'(1) = \pi\gamma(S-s)$.

Next, let

$$M = 1 + \sum_{t=1}^{\infty} M_t \tag{27}$$

denote the total size of productions taken during the adjustment process. We consider the probability that the inventory adjustment process terminates after a finite number of steps, that is, $P(M < \infty)$.

Lemma 3.1. *If $\pi\gamma(S-s) \leq 1$, then $P(M < \infty) = 1$.*

Proof. See Appendix A.1. □

Lemma 3.1 can be considered as the sufficient condition for the existence of competitive equilibrium satisfying Definition 2.1. If $\pi\gamma(S-s) \leq 1$, the adjustment process almost surely terminates in finitely many steps. Otherwise, there is the possibility that the adjustment process never terminates. To be precise, even in the case $\pi\gamma(S-s) > 1$, the adjustment process may terminate, but this is not certain. This includes the case that contradicts Definition 2.1 (iv).

The sequence $\{M_t\}_{t=1}^{\infty}$ is stochastic so that M gives a probability distribution on $\mathbb{N} = \{0, 1, 2, \cdots\}$. We define the probability generating function of $p(m) = P(M = m)$ as follows:

$$g(z) = E[z^M] = \sum_{m=0}^{\infty} p(m) z^m \tag{28}$$

The next lemma provides a key role in evaluating statistical properties of M.

Lemma 3.2. *For $\pi\gamma(S-s) \leq 1$, the generating function $g(z)$ of M satisfies the following equation:*

$$g(z) = z f(g(z)) \ . \tag{29}$$

Proof. See Appendix A.2. □

By using lemma 3.1, we are able to derive the mean of M.

Lemma 3.3. *For $\pi\gamma(S-s) < 1$, the mean of the total size M is finite and given by*

$$E[M] = \frac{1}{1 - \pi\gamma(S-s)} \ . \tag{30}$$

Proof of Lemma 3.3. Differentiating (29), we obtain $g'(z) = f(g(z)) + z f'(g(z)) g'(z)$. Rearranging and considering $z = 1$ leads to $E[M] = g'(1) = 1/(1 - f'(1)) = 1/(1 - \pi\gamma(S-s))$, which has a finite value when $\pi\gamma(S-s) < 1$. □

4 Properties of the equilibrium

4.1 Stationary Distribution

In this section, we examine the properties of the equilibrium and the aggregate fluctuations. First, we investigate the stationary distribution in the long-run equilibrium, as it is called a *statistical equilibrium*. Let π denote the state probability that the stable firm is at the critical level of inventory. We notice that there is no unstable firm in the equilibrium, and hence, the probability that the stable firm is not at the critical level can be denoted by $1-\pi$. In the short-run equilibrium, the actual output, y_n, is obtained under given π. In the long run, however, the probability distribution $(\pi, 1-\pi)$ would vary according to the average of inventory levels changes. The state probability π would increase when the average level of inventories decreases. In contrast, π would decrease when the average level increases.

The average level of inventory is affected by exogenous demands and derived demands caused by inventory adjustments. One unit of exogenous demand makes the firm become unstable with probability π. Once a firm becomes unstable and the inventory adjustment process starts, it is expected that the total amount of inflows of inventory during the process is $(S-s)E[M]$ and the total amount of outflows is $\gamma(S-s)E[M]$. Then, the dynamics of π can be written by

$$\frac{d\pi}{dt} = -\beta\Big[-1+\pi\Big\{(S-s)-\gamma(S-s)\Big\}E[M]\Big] \tag{31}$$

where the parameter β represents the speed of adjustment. The statistical equilibrium $(\pi^*, 1-\pi^*)$ corresponds to the case $d\pi/dt = 0$.

Proposition 4.1. *If the labor share is positive, $\gamma < 1$, then there is a unique stationary distribution given by*

$$\pi^* = \frac{1}{S-s}\ . \tag{32}$$

Proof of Proposition 4.1. For $\pi\gamma(S-s) < 1$, by combining (30) with (31), we have

$$\pi \gtreqqless \frac{1}{S-s} \quad \Leftrightarrow \quad \frac{d\pi}{dt} \lesseqqgtr 0\ .$$

This implies that there is a unique and stable stationary distribution $\pi^* = 1/(S-s)$. In this case, $1-\pi^*\gamma(S-s) = 1-\gamma > 0$. □

Notice that, if $\gamma = 1$, the expected inflows and outflows are equal, and thus, the external perturbation cannot be absorbed by inventory adjustment process. Following is a straightforward corollary of Proposition 4.1 and Lemma 3.1.

Corollary 4.1. *If the labor share is positive, $\gamma < 1$, then there exists an competitive equilibrium satisfying statistical equilibrium almost surely.*

Proposition 4.1 also allows us to use the Poisson approximation, that is, when $\pi\gamma(S-s) = \gamma\ (>0)$,

$$p(k) = P(\xi = k) = \frac{\gamma^k e^{-\gamma}}{k!}, \quad (k = 0, 1, 2, \cdots). \tag{33}$$

The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed. Therefore, it can be used as an approximation of the binomial distribution if $\gamma(S-s)$ is sufficiently large and π is sufficiently small. Under the Poisson approximation (33), the generation function of ξ is given by

$$f(z) = E[z^{\xi}] = \sum_{k=0}^{\infty} \left(\frac{\gamma^k e^{-\gamma}}{k!} \right) z^k = e^{-\gamma(1-z)} \,. \tag{34}$$

It yields both $E[\xi] = f'(1) = \gamma$ and $Var[\xi] = f''(1) + f'(1)(1 - f'(1)) = \gamma$. The mean and variance of M can be calculated as below.

Proposition 4.2. *For $\gamma < 1$, both mean and variance of M are finite and given, respectively, by*

$$E[M] = \frac{1}{1-\gamma} \quad \textit{and} \quad Var[M] = \frac{\gamma}{(1-\gamma)^3} \,.$$

Proof of Proposition 4.2. From Lemma 3.2 and (34), $E[M] = g'(1) = 1/(1 - f'(1)) = 1/(1-\gamma)$. Similarly, $Var[M] = g''(1) + g'(1)(1 - g'(1)) = (f''(1) + f'(1) - f'(1)^2)/(1 - f'(1))^3 = \gamma/(1-\gamma)^3$. □

4.2 Aggregate Fluctuation

The main purpose of this paper is to show how the aggregate fluctuations behave in the statistical equilibrium. In particular, we are interested in testing whether or not the output gap disappears if there is a large number of firms. We firstly express the actual output in terms of M. Let $\boldsymbol{m}(\boldsymbol{\theta})$ be the vector that represent the number of productions corresponding to the demand share vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_n)$. When n is sufficiently large, we can eliminate the possibility that the chain of productions form loops. Then, it seems natural to assume that $\boldsymbol{m}(\boldsymbol{\theta}) = \theta_1 \boldsymbol{m}(\boldsymbol{e}_1) + \theta_2 \boldsymbol{m}(\boldsymbol{e}_2) + \cdots + \theta_n \boldsymbol{m}(\boldsymbol{e}_n)$, where $\boldsymbol{e}_j$ is the j-th unit vector. For any given distribution $(\pi, 1-\pi)$, $\sum_{j=1}^{n} m_j(\boldsymbol{e}_i) = \pi M + (1-\pi)(0)$ is held in the short-run equilibrium. Then the actual output can be expressed by the weighted sum of M_i,

$$y_n = (1-\gamma)(S-s) \sum_{i=1}^{n} m_i(y^* \boldsymbol{\theta}) = \pi(1-\gamma)(S-s) y^* \sum_{i=1}^{n} \theta_i M_i.$$

Consequently, the output gap in the short-run equilibrium can be represented as

$$\log \left(\frac{y_n}{y^*} \right) = \log \left(\pi(1-\gamma)(S-s\ ) \sum_{i=1}^{n} \theta_i M_i \right) . \tag{35}$$

The following theorem is the first main result of this paper.

Theorem 4.1. *We suppose that the labor share is positive, $\gamma < 1$, and the economy is in the statistical equilibrium. For arbitrary demand share, $(\theta_1, \theta_2, \cdots, \theta_n)$, a sequence of output gap converges to zero almost surely, that is,*

$$\log \left(\frac{y_n}{y^*} \right) \quad \xrightarrow{a.s.} \quad 0 \,.$$

Next, we use two lemmas in proving Theorem 4.1.

Lemma 4.1 (Kolmogorov's strong law of large numbers). *Let $\{X_k\}_{k=1}^n$ be a sequence of independent random variables, with finite expectations. If the variance of X_k is finite for each k, and*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} Var[X_k] < \infty \ , \tag{36}$$

then

$$P\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_k - E[X_k]) = 0 \right) = 1 \ . \tag{37}$$

Lemma 4.2 (Limit comparison test). *Suppose that we have two series $\Sigma_k a_k$ and $\Sigma_k b_k$ with $a_k, b_k \geq 0$ for all k. If $\lim_{k\to\infty}(a_k/b_k) = c$ with $0 < c < \infty$, then both series either converge or diverge.*

Proof of Theorem 4.1. Because $\{\theta_i\}$ are given before making a decision, each θ_i and M_i are independent. From Proposition 4.2, both mean and variance of $\theta_i M_i$ are finite.

Recall that $\exp(\varepsilon_i) = n\theta_i$, and hence,

$$\sum_{i=1}^{n} \theta_i M_i = \frac{1}{n} \sum_{i=1}^{n} \exp(\varepsilon_i) M_i \ . \tag{38}$$

We set $a_k = \exp(\varepsilon_k)^2/k^2 > 0$ and $b_k = 1/k^2 > 0$. Without loss of generality, we rearrange $\{\varepsilon_k\}_{k=1}^n$ in descending order, that is, $n > \varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_n \geq \underline{\varepsilon} > 0$. Since $\{\exp(\varepsilon_k)^2\}$ is decreasing and bounded by its infimum $\exp(\underline{\varepsilon})^2 > 0$, the sequence $\{a_k/b_k\}$ is convergent and the limit is given as follows:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{n\to\infty} \frac{\exp(\varepsilon_k)^2/k^2}{1/k^2} = \lim_{k\to\infty} \exp(\varepsilon_k)^2 = \exp(\underline{\varepsilon})^2 > 0.$$

On the other hand, series $\sum_k b_k$ has the limit

$$\lim_{n\to\infty} \sum_k b_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} < \lim_{n\to\infty} \left(1 + \int_1^n \frac{1}{x^2} dx \right) = 2.$$

By Lemma 4.2, the series $\sum_k a_k = \sum_k \exp(\varepsilon_k)^2/k^2$ converges and we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} Var[\exp(\varepsilon_i) M_i] = \frac{\gamma}{(1-\gamma)^3} \sum_{k=1}^{\infty} \frac{\exp(\varepsilon_k)^2}{k^2} < \infty \ . \tag{39}$$

Now, applying Lemma 4.1 to (38), we have

$$P\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Big(\exp(\varepsilon_i) M_i - E[\exp(\varepsilon_i) M_i] \Big) = 0 \right) = 1. \tag{40}$$

Since $\exp(\varepsilon_i) = n\theta_i$ and M_i are independent, we have

$$P\left(\lim_{n\to\infty} \sum_{i=1}^{n} \theta_i M_i = E[M] \right) = 1.$$

Using (32) and Proposition 4.2, we conclude that

$$P\left(\lim_{n\to\infty}\log\left(\frac{y_n}{y^*}\right)=\log(1)\right)=1, \tag{41}$$

which states that the output gap converges to zero with probability 1. □

Theorem 4.1 states that the aggregate fluctuations disappear in the limit $n \to \infty$. This result is in complete contrast to that of Bak et al. (1993), who demonstrate that small shocks can create large fluctuations in aggregate level. The difference between the two arise from the role of labor share. In Bak et al. (1993), the results rely on the specific structure of network and the production technology. It is possible to consider their model as a special case of our model in which the labor share equals to zero, $\gamma = 1$. In fact, even in the case $\gamma = 1$, the sufficient condition for the existence of (short-run) competitive equilibrium is satisfied, so that a competitive equilibrium exists. However, $\gamma = 1$ violates the condition for (long-run) statistical equilibrium stated in Corollary 4.1. Both mean and variance diverge to infinity. The expected inflows and outflows of inventory are equal so that the external perturbation caused by demands cannot be absorbed by the inventory adjustment process. The energy of adjustment is conserved in the very long time periods, generating the large fluctuations of aggregate output. The assumption of zero labor share seems to be special. As we have already seen, for $\gamma < 1$, which seems more realistic, aggregate fluctuations disappear.

4.3 Impulse Response

In order to give a closer look to this point, we consider the stochastic impulse response to exogenous perturbations. We suppose that the expenditure share for good i increases by $d\theta_i$. We notice that the decrease in each θ_j to compensate for the increase in θ_i can be ignored in the large economy. Differentiating (35) with respect to θ_i in the statistical equilibrium, we have

$$\frac{d\log(y_n/y^*)}{d\theta_i}=(1-\gamma)M \propto M\ . \tag{42}$$

The total size of response to the exogenous perturbation is proportional to M. We refer to this as the *stochastic impulse response* to the demand shock for goods i. Therefore, the probability density function of impulse response is proportional to the probability density function of M,

$$p(g)=P\left(\frac{d\log(y_n/y^*)}{d\theta_i}=g\right)\sim P(M=m) \tag{43}$$

The total size M is a random variable. Then, the next proposition provides the probability distribution $p(m) = P(M = m)$ for the case $\gamma \le 1$, which was first studied by Otter (1949).

Proposition 4.3. *We suppose that the labor share is non-negative, $\gamma \le 1$. The probability distribution of total size M satisfies Borel distribution,*

$$P(M=m)=\frac{1}{m}\cdot\frac{(m\gamma)^{m-1}}{(m-1)!}\,e^{-(m\gamma)} \qquad (m=1,2,3,\cdots). \tag{44}$$

Proof of Proposition 4.3. Differentiating (29), we have $f(g(z)) = g'(z)[1 - zf'(g(z))]$. Plugging this into (29) yields

$$g(z) = zf(g(z)) = \sum_{m=0}^{\infty} m \cdot p(m)[1 - zf'(g(z))]z^m \,. \tag{45}$$

Comparing (28) and (45), we obtain $1 = m[1 - zf'(g(z))]$.

The inverse Z-transform of (28) and the Cauchy integral theorem give $p(m)$ as follows:

$$\begin{aligned}\frac{1}{2\pi i}\oint_C g(z)z^{-m-1}dz &= \frac{1}{2\pi i}\oint_C \sum_{k=0}^{\infty} p(k)z^{k-m-1}dz \\ &= \sum_{k=0}^{\infty} p(k)\frac{1}{2\pi i}\oint_C z^{k-m-1}dz = p(m)\end{aligned}$$

where $\oint_C$ denotes the contour integration along a circle surrounding the origin. Here, we set $\omega = g(z)$, and hence,

$$\begin{aligned} p(m) &= \frac{1}{2\pi i}\oint_C \frac{\omega}{(\omega/f(\omega))^{m+1}}dz \\ &= \frac{1}{2\pi i}\oint_C \frac{f(\omega)^m}{\omega^m} f(\omega)dz \\ &= \frac{1}{2\pi i}\oint_C \frac{f(\omega)^m}{\omega^m}[1 - zf'(\omega)]d\omega \\ &= \frac{1}{m}\cdot\frac{1}{2\pi i}\oint_C \frac{[f(\omega)]^m}{\omega^m}d\omega \,. \end{aligned} \tag{46}$$

In the case of Poisson distribution of ξ, from (34), we have

$$f(z)^m = e^{-\gamma m(1-z)} = \sum_{k=0}^{\infty}\left(e^{-(m\gamma)}\frac{(m\gamma)^k}{k!}\right)z^k. \tag{47}$$

Substituting (47) for (46) and using the Cauchy integral theorem, we obtain

$$\begin{aligned} p(m) &= \frac{1}{m}\cdot\frac{1}{2\pi i}\oint_C \sum_{k=0}^{\infty}\left(e^{-(m\gamma)}\frac{(m\gamma)^k}{k!}\right)\frac{\omega^k}{\omega^m}d\omega \\ &= \frac{1}{m}\cdot\frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(e^{-(m\gamma)}\frac{(m\gamma)^k}{k!}\right)\oint_C \omega^{k-m}d\omega \\ &= \frac{1}{m}\cdot\frac{(m\gamma)^{m-1}}{(m-1)!}e^{-(m\gamma)} \,. \end{aligned} \tag{48}$$

Then, we conclude the proposition. □

The probability density function $p(g)$ is proportional to $p(m)$. The following theorem is the second main result in this paper.

Theorem 4.2. *In the competitive equilibrium, the probability density function of impulse response to the demand shock for goods i is analytically given as*

$$p(g) \sim \begin{cases} g^{-\frac{3}{2}}\cdot\exp\left(\dfrac{-g}{\zeta(\gamma)}\right) & \text{if } \gamma < 1 \\ g^{-\frac{3}{2}} & \text{if } \gamma \to 1 \end{cases} \tag{49}$$

where correlation length is given by

$$\zeta(\gamma) = \frac{1}{\gamma - 1 - \ln\gamma}\,. \tag{50}$$

Proof of Theorem 4.2. From Lemma 4.3,

$$p(g) \sim \frac{1}{g}\cdot\frac{(g\gamma)^{g-1}}{(g-1)!}\,e^{-(g\gamma)} = \frac{(g\gamma)^{g-1}}{g!}\,e^{-(g\gamma)}\,.$$

By Stirling's formula $n! \sim \sqrt{2\pi n}\,(n/e)^n$, we obtain

$$p(g) \sim \frac{\left(\gamma e^{(1-\gamma)}\right)^g}{\sqrt{2\pi}\gamma}\cdot g^{-\frac{3}{2}} = \frac{\gamma^{-1}}{\sqrt{2\pi}}\cdot g^{-\frac{3}{2}}\cdot e^{(1-\gamma+\ln\gamma)g} \tag{51}$$

which satisfies the conclusion of the theorem. □

We notice that the correlation length (50) is related to the labor share. The probability density function (49) is simply divided into two classes. In the case that the labor share is positive, $\gamma < 1$, the correlation length is finite. For g smaller than $\zeta(\gamma)$, $p(g)$ is well approximated by the power law with exponent $-3/2$. However, for large g, the exponential decay dominates the power law decay. The resulting distribution has an exponential tail, with a characteristic scale given by the correlation length. The impulse response finitely converges. The probability of having the propagation size larger than $\zeta(\gamma)$ is extremely small. This means that the effect of idiosyncratic shocks get extinct early, and could not propagate larger than the scale given by (50). The inventory adjustment process in no way can propagate the micro-level lumpiness larger than the certain scale.

However, precisely in the case that the labor share is zero, $\gamma = 1$, the correlation length (50) diverges to infinity. This means that the critical point of this economy is the case where the labor share is zero. Then, the exponential part disappears and the distribution exhibits a power law decay even for sufficiently large g. The size of the impulse response diverges so that the effect of shocks cannot be absorbed and conserved in the economic system. Since the characteristic scale is absent at the critical point, huge propagations are possible. However, as we have already mentioned, it seems unreasonable to assume that the labor share is zero.

5 Conclusion

This paper investigates the relationship between lumpy inventory adjustment and aggregate fluctuations. We present a general equilibrium model composed of many firms interacting within the input–output network. The main feature of the model is that each firm follows the (S, s) inventory policy and the production processes in any firm make use of input from any other firms in the economy. The (S, s) inventory policy leads the firm to lumpy production pattern and the interaction between firms can create the chain of productions. It is possible that fluctuations caused by micro-level lumpiness propagate over the input–output linkages across different firms within the economy.

The contributions of this paper are as follows. First, we provide a tractable method to compute the aggregate output in a general-equilibrium framework with (S, s) inventory policies. We characterize the competitive equilibrium and give the closed form solution. Because of the complexity of the lumpy adjustment process, it is hard to determine the exact amount of micro-level quantities. Then, we employ the branching process to estimate the key statistics of inventory adjustment process. We define the operator that represents the inventory adjustment caused by single production and describe the chain of productions by sequentially applying these operators. We use the statistics of this sequence instead of the exact calculation of the adjustment process. Using these estimates, we derive the stationary distribution in the statistical equilibrium. The competitive equilibrium restricts the short-run equilibrium when the distribution of inventory level across firms is given. In the long run, the economy reaches the stationary state in distribution. We prove that the stationary distribution of this economy can be determined by the (S, s) band under the homogeneity assumption.

Second, we show that the strong law of large numbers holds in our framework. We prove that when the network structure is symmetric and all firms are homogeneous, the output gap converges to zero almost surely in the statistical equilibrium. In other words, if the economy is sufficiently large, the aggregate inventory fluctuations do not solely arise from a combination of micro-level lumpiness and economic interactions among firms. This implies that the micro-level lumpiness has negligible impact on the macro-level activity in this framework.

We also show that the positive labor share reduces the high fluctuations in aggregate inventory level and moderates the business cycle. This can be understood from the viewpoint of impulse response to demand shock. If the labor share is positive, the probability density function of size of impulse response has an exponential tail. The inventory adjustment process cannot propagate the micro-level lumpiness larger than the certain scale. Further, only in the limit case where the labor share is zero, the exponential part disappears and the distribution exhibits a power law decay.

From the results of this paper, we can conclude that the criticality hypothesis does not hold by itself in our framework. In particular, if labor share is positive, system criticality does not solely arise from a combination of lumpiness, attributed to non-convexity, and firms' local interactions, without network or granular effects. Both network and granular effects are essential sources of large fluctuations. Nevertheless, this does not imply that non-convexity is a meaningless factor in aggregate fluctuations. If a network structure is asymmetric or firm sizes are heterogeneous, the effect of lumpiness from non-convexity may not be negligible in macro-level activities. In such situations, it seems possible that the amplification mechanism of both network and granular effects can be strengthened by micro-level lumpiness in the inventory adjustment process.

A Appendix

A.1 Proof of Lemma 3.1

Proof. The inventory adjustment process terminates when $M_t = 0$. Then, all the subsequent M_t are also zero. The probability that the adjustment process terminate until step T is given by

$$q^{(T)} = P\left(\bigcup_{t=1}^{T}\{M_t = 0\}\right) \in [0,1] \,. \tag{52}$$

Note that $\{M_t = 0\} \subseteq \{M_{t+1} = 0\}$ for all $t = \{1, 2, \cdots\}$. The probability that $q^{(T)}$ is increasing in T and is bounded above by 1, and then, $q = \lim_{T\to\infty} q^{(T)}$ exists.

Under Assumption 2.1, we can consider M_{t+1} as the sum of independent copies of M_t. We suppose that $M_1 = k$, and then, we have

$$E[z^{M_{t+1}} \mid M_1 = k] = E[z^{M_t^{(1)} + M_t^{(2)} + \cdots M_t^{(k)}}] = (E[z^{M_t}])^k \tag{53}$$

Let $f_t(z) = E[z^{M_t}]$ denote the probability generating function of M_t. Plugging (53) into $f_{t+1}(z)$, we get the following recursion:

$$\begin{aligned} f_{t+1}(z) = E[z^{M_{t+1}}] &= \sum_{k=0}^{\infty} E[z^{M_{t+1}} \mid M_1 = k] P(M_1 = k) \\ &= \sum_{k=0}^{\infty} (f_t(z))^k p_k \\ &= f(f_t(z)) \end{aligned} \tag{54}$$

Since $f_t(0) = P(M_t = 0) = q^{(t)}$, then $q^{(t+1)} = f(q^{(t)})$. We obtain $q = f(q)$ by taking $t \to \infty$. Therefore, q is the attractive fixed point of the map $z \mapsto f(z)$.

From definition, $f(1) = 1$. In this model, $0 < p_k < 1$ for all $k \geq 0$, and then, $f(0) = p_0 > 0$. By differentiating $f(z)$ repeatedly, we have $f'(z) > 0$ and $f''(z) > 0$, which implies that $f(z)$ is increasing and strictly convex for $z \in [0, 1]$. If $f'(1) = \pi\gamma(S - s) \leq 1$, the map $z \to f(z)$ has a unique fixed point $z^* = 1$. If $f'(1) = \pi\gamma(S - s) > 1$, the map $z \to f(z)$ has two fixed points and the smallest one is attractive: $z^* = f(z^*)$ with $z^* < 1$. These results are summarized by

$$q = \begin{cases} 1 & \text{if } \pi\gamma(S - s) \leq 1 \\ q^* & \text{if } \pi\gamma(S - s) > 1 \end{cases} \tag{55}$$

with $q^* < 1$. □

A.2 Proof of Lemma 3.2

Proof. (See also Harris(1963) Theorem 8.2.) Suppose the number of productions in $t = 1$ is given by k, then the total number M can be decomposed into $M = 1 + M^{(1)} + M^{(2)} + \cdots + M^{(k)}$. The $M^{(j)}$ are independent and have the same distribution as M. Thus, we have

$$\begin{aligned} E(z^M \mid M_1 = k) &= E(z^{1+M^{(1)}+M^{(2)}+\cdots+M^{(k)}}) \\ &= zE(z^M)E(z^M)\cdots E(z^M) \\ &= zg(z)^k. \end{aligned} \tag{56}$$

From this,

$$\begin{aligned} g(z) = E(z^M) &= \sum_{k=0}^{\infty} E(z^M \mid M_1 = k) P(M_1 = k) \\ &= z \sum_{k=0}^{\infty} p_k [g(z)]^k \\ &= z f(g(z)). \end{aligned} \tag{57}$$

□

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