Stochastic multiplier in the (S, s) economy

Naoto Yagi *†

This Version: December 8, 2015

Abstract

This paper proposes a new multiplier process model with discrete inventory adjustment and input-output structure. The (S, s) inventory policy and a regular network are employed. We define the multiplier process as a branching and derive a statistics of the multiplier in the statistical steady state. We show that convergence and termination of the multiplier process are different matters in a statistical model. We prove that the sufficient condition for convergence of expected multiplier is the same as the Brauer–Solow condition, which is a sufficient condition for a linear input-output model. We also prove that the necessary and sufficient condition for termination of a multiplier process is weaker than the Hawkins–Simon condition, which is a necessary and sufficient condition for the linear model. We examine the relation between the rate of return and the multiplier. We prove that the multiplier is finite and its probability distribution function decays exponentially if the rate of return is sufficiently large, whereas the multiplier asymptotically diverges and the probability distribution function is asymptotically a power law if the rate of return is close to zero. This indicates the relation between low profit rate and high output volatility.

JEL Classification: C60, E32
Keywords: Multiplier, (S, s) Economy, Inventory Dynamics, Input-output Structure, Endogenous Business Cycle

1 Introduction

Inventory adjustment has been assumed to play an important role in multiplier processes ever since the concept was introduced by Richard Kahn and John Maynard Keynes in the 1930s. During a multiplier process, the aggregate demand is higher than the aggregate output and the total inventory in the economy continues to fall. To prevent the total inventory from falling to an undesirably low level, the firm must produce some extra output and restore its inventory

*Department of Economics and Business Administration, Niigata Sangyo University.
†E-mail : yagi@econ.nsu.ac.jp

levels during the multiplier process. The firm's decision on how and when to do so would affect the economy's reaction to disequilibrium and its propagation mechanism. There has been considerable research on linear and continuous adjustment models. Nevertheless, when it comes to the discrete adjustment model, it is still unclear as to how the inventory adjustment process moves the economy to a new equilibrium after the multiplier process.

In inventory theory, the (S, s) policy is a well-known adjustment effective in situations involving non-convex adjustment costs. Arrow, Harris, and Marschak (1951) introduced the (S, s) form of inventory policy, while a general proof of the optimality of these policies was provided by Scarf (1960). We describe the characteristics of the (S, s) policy below. The firm establishes a lower stock point s and an upper stock point S. If the inventory level is above the lower point, the firm satisfies the requirement of inventory and no production takes place. If the inventory level falls to or below the point s, the firm produces a certain amount of goods in order to restore the inventory level to the upper point S.

This class of economies, called the (S, s) economy, has attracted a large number of researchers concentrating on its aggregate consequences. For a study of macroeconomic impact, we need to aggregate the individual policy. However, this is a difficult task because of the nonlinearity and discreteness of the (S, s) policy due to its non-convex environment. Caplin (1985) considered the aggregate implication of exogenous (S, s) inventory policies across firms that follow a time-invariant policy. Using a Markov process model, he concluded that in the long run, the inventory levels of individual retailers are mutually independent regardless of correlation in sales. Caballero and Engle (1991) provide a framework for analysis of the aggregate dynamics of the (S, s) economy and give the conditions under which the (S, s) economy achieves a steady state; here, the steady state is defined as a condition in which the distribution of inventories is invariant to the distribution of demand.

Bak, Chen, Scheinkman, and Woodford (1993), hereafter BCSW, built a production and inventory economy model with large endogenous fluctuations. They demonstrated that random shocks to an economy do not average out in the aggregate but might produce significant aggregate fluctuations. They incorporate local interactions through production networks into the (S, s) economy; these are absent in Caplin (1985) and Caballero and Engle (1991). BCSW used the *sandpile model* (Bak et al. (1989)) and considered a simple square economy. They assumed that firms connect through a hierarchical network involving some amount of inventory. This mechanism demonstrated that random shocks to the economy do not average out in the aggregate but might produce significant fluctuations. Although the BCSW model is very attractive, it depended heavily on a particular kind of production network, that is, the hierarchy of order flow.

In this paper, we generalize the model and study a more general pattern of connections between firms and the multiplier process with the (S, s) discrete adjustment economic policy. The rest of the paper is organized as follows. The next section presents the model we use and defines the multiplier process, including the discrete inventory adjustment process. In section 3, we characterize the statistical property of the model. Section 4 proves four key theorems, and section 5 concludes the paper.

2 The model

2.1 General framework of the (S, s) economy

In an economy consisting of a large number of firms, assume that $\mathcal{N} = \{1, 2, \cdots, n, \cdots, N\}$ is a set of firms. Each firm produces differentiated goods indexed by $i \in \mathcal{N}$. Further, integer x_i represents the number of inventories held by firm i, and a configuration of inventory holdings $\boldsymbol{x'} = (x_1, \cdots, x_N)$ is a vector of non-negative integers.

Each firm i sets a pair of integers (S_i, s_i) with $S_i > s_i + 1$ so as to minimize costs, where S_i is the upper target of inventory level and s_i, the lower point. Two positive integers, (S_i, s_i), specify the inventory policy of firm i. If the inventory level falls to s_i or below, that is, $x_i \leq s_i$, we say that firm i is *unstable* with regard to inventory. Similarly, if $s_i < x_i$, firm i is *stable*. A stable configuration indicates stability at all firms. We denote the set of all stable configurations by $\mathcal{S}$; that is,

$$\mathcal{S} = \{(x_1, \cdots, x_n) \in \mathbb{Z}_+^n \mid S_i \geq x_i > s_i, \forall i \in \mathcal{N}\}. \tag{1}$$

We further assume that (S_i, s_i) is time-invariant for all firms, so that stable set $\mathcal{S}$ is also time-invariant. All firms monitor their inventory levels in order to avoid costly stock-outs each period and maintain the stability of their inventory levels. When a firm becomes unstable, it needs to increase its production immediately and restore its target level.

Similarly, (S_i, s_i) determines its production lot size, $\Delta_i = S_i - s_i > 1$, which indicates the amount firm i can produce in a period. Whether or not a firm produces goods depends on the level of inventory the firm has at the beginning of the period and the order for that period. On receiving an order, the firm satisfies its inventory position. If the firm is stable after execution of the order, no need exists for further production. If firm i becomes unstable following the order, it must produce $\Delta_i = S_i - s_i$ units of goods in that period.

In this economy, firms are interdependent in terms of input. Every firm needs to procure certain units of output from other firms. For its production, firm j needs $(\Delta_{1j}, \cdots, \Delta_{ij}, \cdots, \Delta_{Nj})$ of goods produced by other firms (notice that these inputs include firm j's own products, Δ_{jj}). One production by firm j means that its inventory level increases by $\Delta_j = S_i - s_i$, while the inventory levels of all the associated firms decrease by Δ_{ij}. Besides the assumption of production lot size Δ_j, we assume that the vectors of inputs $(\Delta_{1j}, \cdots, \Delta_{ij}, \cdots, \Delta_{Nj}), \forall j \in \mathcal{N}$ are time-invariant.

We define the non-negative integer m_i as the amount of production taking place at firm i during the given period and note that vector $\boldsymbol{m'} = (m_1, \cdots, m_N) \in \mathbb{Z}_+^N$. Let integer $y_i \geq 0$ denote the order as an exogenous demand on firm i. Furthermore, let x_i and x'_i denote respectively firm i's *ex-ante* and *ex-post* inventory levels. The following equations represent the inventory dynamics of the economy:

$$x'_i = x_i - y_i + m_i\Delta_i - \sum_{j \in \mathcal{N}} m_j \Delta_{ij}, \quad \forall i \in \mathcal{N}. \tag{2}$$

The term $m_i\Delta_i$ represents the total amount of products firm i produces during the given period. This term has a positive role in inventory accumulation. Term $m_j\Delta_{ij}$ represents the total amount of inputs moved from i to j for the production of firm j. This term has a negative, or at least non-positive, role in inventory accumulation.

We consider the equations of inventory dynamics as the equation system representing the structure of the (S, s) economy.

$$\begin{pmatrix} m_1\Delta_1 \\ m_2\Delta_2 \\ \vdots \\ m_N\Delta_N \end{pmatrix} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x'_1 - x_1 \\ x'_2 - x_2 \\ \vdots \\ x'_N - x_N \end{pmatrix} .$$

Matrix $D = (\Delta_{ij})_{i,j\in\mathcal{N}}$ is an $N \times N$ adjacency matrix of a weighted graph representing the input network between firms.

Now, the model has been fully described. Following Caplin (1985), Definition 2.1 summarizes the components of the model below.

Definition 2.1. ***The*** (S, s) ***economy*** *consists of*

- *a set of firms:* $\mathcal{N} = \{1, 2, \cdots, n, \cdots, N\}$
- *N pairs of (S, s) policies:* $\{\Delta_i\}_{i\in\mathcal{N}}$ *with* $\Delta_i = S_i - s_i > 1$
- *an* $N \times N$ *adjacency matrix of a weighted graph representing the input network of the economy:* $D = (\Delta_{ij})_{i,j\in\mathcal{N}}$
- *an exogenous demand vector:* $\boldsymbol{y} = (y_1, \cdots, y_N) \in \mathbb{Z}_+^N$
- *transition equations of inventory level given by (2).*

We define the input coefficient as

$$a_{ij} = \frac{\Delta_{ij}}{\Delta_i} \in \mathbb{Q}_+ \ , \tag{3}$$

and the input coefficient matrix as $A = (a_{ij})_{i,j\in\mathcal{N}}$. All Δ_i are positive integers and all Δ_{ij} are non-negative; therefore, input coefficients a_{ij} are non-negative rational numbers. Since both Δ_i and Δ_{ij} are time-invariant, the input coefficients a_{ij} are constant. From a_{ij}, we obtain

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & 1-a_{NN} \end{pmatrix} \begin{pmatrix} m_1\Delta_1 \\ m_2\Delta_2 \\ \vdots \\ m_N\Delta_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} + \begin{pmatrix} x'_1 - x_1 \\ x'_2 - x_2 \\ \vdots \\ x'_N - x_N \end{pmatrix} ,$$

or its matrix form representation

$$\boldsymbol{\Delta}[I - A]\boldsymbol{m} = \boldsymbol{y} + \boldsymbol{x}' - \boldsymbol{x} \ , \tag{4}$$

where A, $\boldsymbol{\Delta}$, and $\boldsymbol{y}$ are the input coefficient matrix, the vector of production lot size, and an exogenous demand vector, respectively.

The problem of the (S, s) economy is to determine $\boldsymbol{m}$ (and hence $\boldsymbol{x}'$) for given $\boldsymbol{\Delta}$, A, and $\boldsymbol{x}$ when the exogenous demand takes a certain vector $\boldsymbol{y}$. In the linear model, as with the Leontief input-output model, we can obtain the solution from the inverse matrix $[I - A]^{-1} = I + A + A^2 + \cdots$, which is called the *matrix multiplier*. However, the adjustment process of the (S, s) economy is non-linear and composed of many (S_i, s_i) policies, which are also non-linear adjustment policies. In addition, individual non-linearity can be amplified through interaction with the input network.

2.2 Multiplier process of the (S, s) economy

To determine $\boldsymbol{m}$ in such a non-linear model, we describe how $\boldsymbol{m}$ can be constructed through the inventory adjustment process. The dynamics is defined as follows. At first, the initial state of inventory holding for each firm would randomly take a configuration $\boldsymbol{x}' = (x_1, \cdots, x_N)$ from the stable set $\mathcal{S}$. Then, we have one unit of exogenous demand occurring at a time to a randomly chosen firm i. This decreases i's inventory level by one. If $x_i \to x_i - 1 < s_i$, $\boldsymbol{x}$ becomes unstable at firm i and the inventory adjustment process starts, but otherwise $\boldsymbol{x}$ will be stable and the next perturbation occurs.

The inventory adjustment process starts with one single production; we call this step $t = 0$ of the process. Let i_0 be the index of this firm, which must update $m_{i_0} = 0 \to m_{i_0} = 1$. At the same time, i_0 needs $(\Delta_{i_0 1}, \cdots, \Delta_{i_0 N})$ of input and derived demand occurs for other firms. This derived demand decreases Δ_{ij} units of inventory of each firm and may render one or more of the firms unstable. Let $\{i_1\}$ be the list of unstable firms at $t = 1$. Firms in $\{i_1\}$ have to produce and increase their inventories within step $t = 1$ and update $m_{i_1} \to m_{i_1} + 1, \forall i_1 \in \{i_1\}$. Similarly, the production in step $t - 1$ may lead to further production in step t. Let $Z_t^{(i)} = m_{i,t} - m_{i,t-1} \in \{0, 1\}$ be the update score of firm i at step t, which is a random variable in a large economy, and let $\boldsymbol{Z}_t = (Z_t^{(1)}, \cdots, Z_t^{(N)})$ be the vector of scores. Furthermore, let $\boldsymbol{m_t}$ be the vector of the total score until t, $\boldsymbol{m}_t$ is generated by random vector $\boldsymbol{Z}_t$.

Definition 2.2. ***A multiplier process in the*** (S, s) ***economy*** *is defined as the stochastic process* $\{\boldsymbol{m}_t\}_{t=0}^{\infty}$ *generated by*

$$\boldsymbol{m}_t = \boldsymbol{m}_{t-1} + \boldsymbol{Z}_t \tag{5}$$

Unfortunately, we find it very difficult to explicitly solve (5) because the (S, s) economy described here is rather general and contains several heterogeneities and irregularities. Inventory policies and input networks are different from firm to firm. Each firm's updating therefore evolves different (S_i, s_i) policies and so $\{Z_t^{(i)}\}$ are not identically distributed. Furthermore, the timing of updating is dependent on other firms and so $\{Z_t^{(i)}\}$ is not independent.

2.3 The n-regular (S, s) economy

We introduce regularities into the economy to make the model more tractable from this point onward.

Assumption 2.1. We assume two regularities in the (S, s) economy.

1. Regularity of (S, s) policy: $\Delta_i = \Delta = S - s > 1, \;\; \forall i \in \mathcal{N}$

2. Regularity of input matrix:

   (a) $\Delta_{ij} \in \{1, 0\}$

   (b) $\sum_j \Delta_{ij} = n, \ \ \forall i \in \mathcal{N}$ and $0 < n << N$.

Assumption 2.1 (1) is about the regularity of (S, s) policy. We assume that all firms have equal (identical) (S, s) policies. This assumption means that every firm has the same lot size and produces the same amount in a single production.

Assumptions 2.1 (2.a) and (2.b) are about the regularity of inputs. We assume that if there is an input relation from i to j, then element $\Delta_{ij} = 1$, otherwise $\Delta_{ij} = 0$. We also assume that the number of inputs the firm needs in a single production of Δ is equal (identical) for all firms. For a given firm, the list of firms from which it gets inputs is fixed. These two assumptions on input regularity suggest that we restrict the graph to a regular graph representing an input network between firms under assumption 2.1. When the number of inputs is equal to n, the graph is referred to as an *n-regular graph*.

Now, under assumption 2.1, we can summarize an n-regular (S, s) economy by definition 2.3.

Definition 2.3. ***An n-regular*** (S, s) ***economy*** *consists of*

- *a set of firms:* $\mathcal{N} = \{1, 2, \cdots, n, \cdots, N\}$
- *identical (S, s) policies for all firms, given by assumption 2.1 (1)*
- *an $N \times N$ adjacency matrix of* ***n-regular graph*** *representing the input network of economy:* $D = (\Delta_{ij})_{i,j\in\mathcal{N}}$, *given by assumptions 2.1 (2.a) and (2.b)*
- *an exogenous demand vector:* $\boldsymbol{y} = (y_1, \cdots, y_N) \in \mathbb{Z}_+^N$
- *transition equations of inventory level given by (2).*

From the perspective of analytic solvability, we benefit from these assumptions. The input coefficients of the n-regular (S, s) economy become identical

$$a_{ij} = \begin{cases} a \ \left(= \frac{1}{\Delta}\right) & \text{if firm } j \text{ needs input from firm } i \\ 0 & \text{otherwize,} \end{cases} . \tag{6}$$

and

$$\sum_j a_{ij} = na, \ \forall i \in \mathcal{N}. \tag{7}$$

Considering the regularities of Δ, we define the total number of productions in step t, Z_t and that until step t, M_t, as

$$Z_t \equiv \sum_{i\in\mathcal{N}} Z_t^{(i)} \tag{8}$$

$$M_t \equiv \sum_{i\in\mathcal{N}} m_i \tag{9}$$

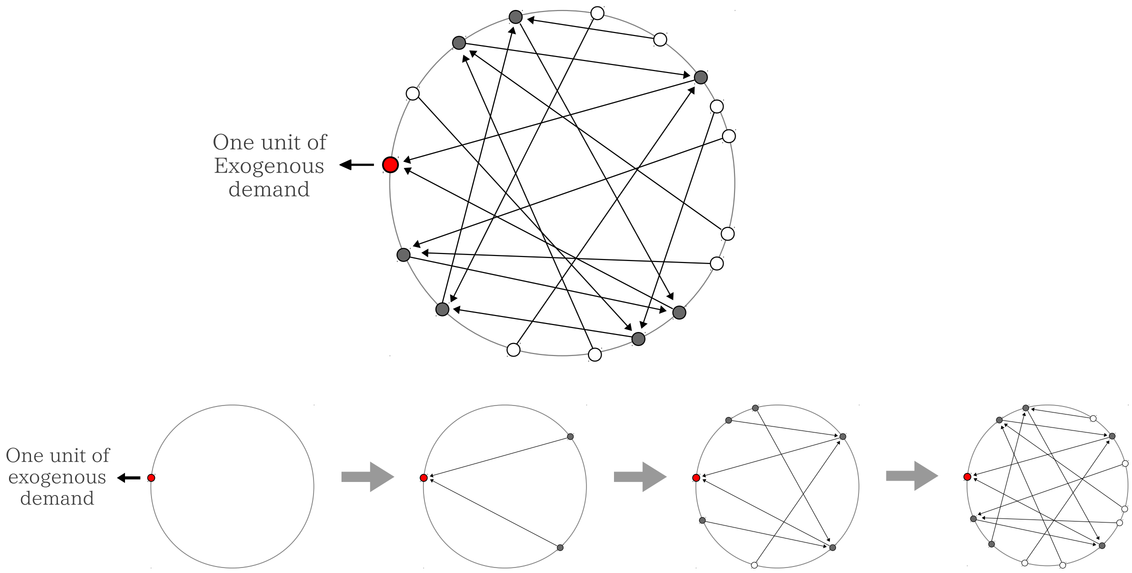

Figure 1: Description of inventory adjustment process in a 2-regular (S, s) economy. Arrows $\longrightarrow$ represent the direction of inputs. Points $\bullet$ represent firms with stock-out and production in process, and points $\circ$ represent firms with sufficient inventories to terminate the chain of production.

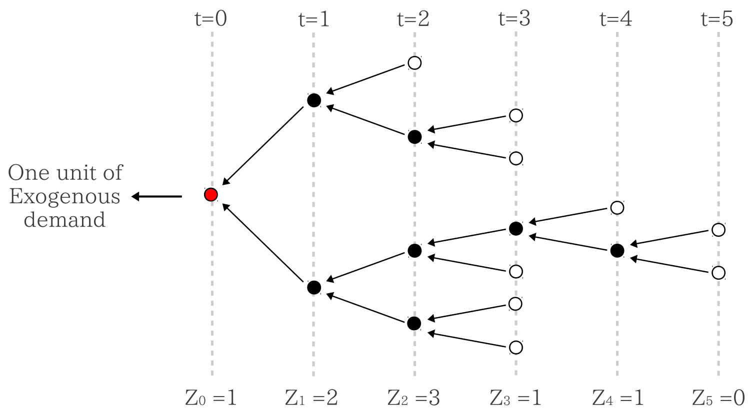

Figure 2: Multiplier process in a 2-regular (S, s) economy. $Z_0 \equiv 1, Z_1 = 2, Z_2 = 3, Z_3 = 1, Z_4 = 1, Z_5 = 0$. $M_5 = Z_0 + Z_1 + Z_2 + Z_4 + Z_5 = 8$.

With the regularity of number of inputs, n, the random variable Z_t can be represented as follows. Let K be the number of firms that start production of new inventories from the derived demand of a single firm. Whether a firm starts production depends on its inventory level. More precisely, K is decided by the ratio of firms whose inventory level becomes $x_i \to x_i - 1 < s_i$ by one order. Let π be the ratio of firms whose inventory level is equal to 1 per N.

$$\pi \equiv \frac{\text{\# of firms whose inventory level is equal to 1}}{N} \ . \tag{10}$$

The probability that a firm that produced once will produce again in the next steps of the same process is of the order of $1/N$. If N is sufficiently large, we can eliminate the possibility that the chain of production forms loops. A binomial distribution gives the probability of k times of production triggered by a single production,

$$p_k = P(K = k) = \left(\begin{array}{c} n \\ k \end{array} \right) \pi^k (1-\pi)^{n-k}, \quad (k = 0, 1, \cdots n) \ . \tag{11}$$

From this nature of K, we can construct the stochastic process $\{Z_t\}_{t=0}^{\infty}$ as follows. We start with $Z_0 = 1$. Z_1 is the number of firms that produced in step $t = 1$. Obviously, $Z_1 = K$. Since Z_t is the number of firms that produced in step t, Z_t is obtained from the recursive equation

$$Z_t = \sum_{j=1}^{Z_{t-1}} K_j \ , \tag{12}$$

where K_i corresponds to the number of firms that produced in step t from the derived demand of firm i, which produced in step $(t-1)$.

From this recursion of Z_t, we define the multiplier process in the n-regular (S, s) economy as follows:

Definition 2.4. ***A multiplier process in the n-regular*** (S, s) ***economy*** *is defined as a stochastic process* $\{M_t\}_{t=0}^{\infty}$*, that is,*

$$M_t = M_{t-1} + Z_t \tag{13}$$

3 Stochastic properties of multiplier process

3.1 Probability generating function

Now, let ϕ be the probability generating function of K as

$$\phi(\theta) = \phi_K(\theta) = E(\theta^K) = \sum_{k=0}^{\infty} p_k \theta^k \ , \quad |\theta| \le 1. \tag{14}$$

We also define the probability generating function of Z_t as

$$\phi_{Z_t}(\theta) = E(\theta^{Z_t}) = \sum_{k=0}^{\infty} P(Z_t = k) \theta^k \ , \quad |\theta| \le 1. \tag{15}$$

Since $K_1, \cdots, K_k$ are independent and $\phi(\theta) = \phi_{Z_1}(\theta) = \phi_K(\theta)$ because $Z_1 = K$, we have

$$\begin{aligned}
\phi_{Z_t}(\theta) &= E(\theta^{Z_t}) \\
&= \sum_{k=0}^{\infty} E\big[\theta^{Z_t} \mid Z_{t-1} = k\big] \cdot P(Z_{t-1} = k) \\
&= \sum_{k=0}^{\infty} E\big[\theta^{K_1 + \cdots + K_k}\big] \cdot P(Z_{t-1} = k) \\
&= \sum_{k=0}^{\infty} E(\theta^{K})^k \cdot P(Z_{t-1} = k) \\
&= \sum_{k=0}^{\infty} \phi(\theta)^k P(Z_{t-1} = k) \ . \qquad (16)
\end{aligned}$$

From the definition of ϕ,

$$\phi_{Z_{t-1}}(\phi(\theta)) = \sum_{k=0}^{\infty} \phi(\theta)^k P(Z_{t-1} = k) \ , \qquad (17)$$

we have

$$\phi_{Z_t}(\theta) = \phi_{Z_{t-1}}(\phi(\theta)) \ . \qquad (18)$$

Thus,

$$\phi_{Z_t}(\theta) = \phi^t(\theta) \qquad (19)$$

3.2 Expected values

Following the binomial theorem, the generating function becomes

$$\phi(\theta) = \sum_{k=0}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} \theta^k = (\pi\theta + (1-\pi))^n \ . \qquad (20)$$

Thus, we have the following lemmas.

Lemma 3.1. $\mathbf{E}(K) = n\pi < \infty$.

Proof. By differentiating (20), we have

$$\phi'(\theta) = \sum_{k=0}^{n} k p_k \theta^{k-1} = n[\pi\theta + (1-\pi)]^{n-1} \cdot \pi$$

Taking the limit $z \to 1-0$, and recalling that $0 < n \leq \Delta$ and $0 \leq \pi \leq 1$, we have

$$\mathbf{E}(K) = \lim_{\theta \to 1-0} \phi'(\theta) = n\pi < \infty.$$

□

Lemma 3.2. $\mathbf{E}(Z_t) = (n\pi)^t < \infty$.

Proof. Differentiating (18), we have

$$\phi'_{Z_t}(\theta) = \phi'_{Z_{t-1}}(\phi(\theta)) \cdot \phi'(\theta). \tag{21}$$

From limit $z \to 1-0$ and lemma 3.1, we have

$$\begin{aligned} \mathbf{E}(Z_t) &= \lim_{\theta\to 1-0} \phi'_{Z_t}(\theta) \\ &= \lim_{\theta\to 1-0} \phi'_{Z_{t-1}}(\phi(\theta)) \cdot \lim_{\theta\to 1-0} \phi'(\theta) \\ &= \mathbf{E}(Z_{t-1}) \cdot \mathbf{E}(K) \\ &= \mathbf{E}(K)^t \\ &= (n\pi)^t \ . \end{aligned}$$

□

The expected number of productions conditioned to the value of previous productions, $\mathbf{E}(Z_{t+1} \mid Z_t = z_t)$, can be obtained, since $Z_t = z_t$ is fixed, from the expectation of $Z_{t+1} = \sum_{i=1}^{z_t} K_i$; that is,

$$\mathbf{E}(Z_{t+1} \mid Z_t = z_t) = \mathbf{E}\left(\sum_{i=1}^{z_t} K_i\right) = z_t \cdot \mathbf{E}(K) \ . \tag{22}$$

4 Multiplier in the statistical equilibrium

4.1 Definition of stochastic multiplier

Before discussing the stochastic multiplier, let us define the statistical steady state of the economy, the so-called *statistical equilibrium*. Following repeated perturbations of exogenous demand, the economy reaches a statistical equilibrium state, when the aggregate inflow of inventories into the economy must be equal to the aggregate outflow of inventories from the economy. Let X_{in} and X_{out} denote the total inflow and outflow of inventories, respectively, following the multiplier process triggered by one exogenous demand; $\mathbf{E}[X_{in}]$ and $\mathbf{E}[X_{out}]$ are respectively the expected values.

Definition 4.1 (statistical equilibrium). *The statistical equilibrium of the n-regular (S,s) economy is defined as a state in which the expected aggregate inflow equals the expected aggregate outflow; that is,*

$$\mathbf{E}[X_{in}] = \mathbf{E}[X_{out}]. \tag{23}$$

Statistical equilibrium can be characterized by a single parameter.

Proposition 4.1. *In statistical equilibrium, $\pi = a$.*

Proof. The inflow at step t can be given by $Z_t\Delta$. From lemma 3.2, the expected value of X_{in} is

$$\mathbf{E}(X_{in}) = \sum_{t=0}^{\infty} \mathbf{E}(Z_t)\Delta = \Delta \sum_{t=0}^{\infty} (n\pi)^t \ . \tag{24}$$

Note that the outflow consists of two parts. An exogenous demand of $1/\pi$ is required before an adjustment process starts. Once the process starts, the outflow at step t can be expressed by $Z_{t-1} \cdot n$. Then, the expected number of X_{out} becomes

$$\mathbf{E}(X_{out}) = \frac{1}{\pi} + \sum_{t=1}^{\infty} \mathbf{E}(Z_{t-1})n = \frac{1}{\pi} + \sum_{t=1}^{\infty} (n\pi)^{t-1} n = \frac{1}{\pi} \sum_{t=0}^{\infty} (n\pi)^t . \tag{25}$$

In the equilibrium $\mathbf{E}(X_{in}) = \mathbf{E}(X_{out})$, we get

$$\pi = \frac{1}{\Delta} = a \tag{26}$$

□

Assume that M denotes the limit of multiplier process $\{M_t\}$,

$$M = \lim_{t\to\infty} M_t . \tag{27}$$

Definition 4.2 (stochastic multiplier). *In statistical equilibrium, a stochastic multiplier of the n-regular (S, s) economy is defined as the expectation of the limit of stochastic process $\{M_t\}$ with probability $\pi = a$; that is,*

$$\mathbf{E}(M) = \mathbf{E}\left(\lim_{t\to\infty} M_t\right) \quad \text{with} \quad \pi = a . \tag{28}$$

4.2 Convergence of stochastic multiplier

Theorem 4.1 (convergence). *In statistical equilibrium, the multiplier process of the n-regular (S, s) economy converges and can be expressed exactly as*

$$\mathbf{E}(M) = \frac{1}{1 - na} = 1 + na + (na)^2 + (na)^3 + \cdots , \tag{29}$$

if the ***Brauer–Solow sufficient condition*** *holds, that is,*

$$\sum_{j=1}^{N} a_{ij} < 1, \quad \forall i \in \mathcal{N} . \tag{30}$$

Proof. For the interchange of limits and expectations and the linearity of expectations, the RHS of (28) can be

$$\begin{aligned}
\mathbf{E}\left(\lim_{t\to\infty} M_t\right) &= \lim_{t\to\infty} \mathbf{E}(M_t) \\
&= \lim_{t\to\infty} \mathbf{E}\left[\sum_{\tau=0}^{t} Z_\tau\right] \\
&= \lim_{t\to\infty} \sum_{\tau=0}^{t} \mathbf{E}(Z_\tau) \\
&= 1 + \mathbf{E}(Z_1) + \mathbf{E}(Z_2) + \mathbf{E}(Z_3) \cdots \\
&= 1 + na + (na)^2 + (na)^3 + \cdots \\
&= \frac{1}{1 - na} \quad (|na| < 1) .
\end{aligned}$$

If (30) holds, $1 > \sum_{i=1}^{N} a_{ij} = na > 0$ for all $i \in \mathcal{N}$. This satisfies the convergence radius of (29). □

4.3 Termination of multiplier process

Next, we examine the probability that the multiplier process terminates in finite steps, or, in another words, the multiplier process contains finite mass; that is,

$$P(M < \infty) = P\left(\lim_{t\to\infty} \sum_{\tau=0}^{t} Z_\tau < \infty\right) = 1 \ . \tag{31}$$

Note that this problem is different from the convergence of multiplier, which we proved in Theorem 4.1. Theorem 4.1 gives the condition for convergence of the "expectation value" of the multiplier. The following theorem gives the condition for convergence of the multiplier "process" itself.

Let ω_t be the probability that the multiplier process terminates by the t-th step,

$$\omega_t = P(Z_t = 0) \ . \tag{32}$$

Further, let ω be the limit, if one exists,

$$\lim_{t\to\infty} \omega_t = \omega. \tag{33}$$

If the multiplier process terminates, $\omega = 1$, otherwise $\omega < 1$.

Theorem 4.2 (termination probability). *In statistical equilibrium, the multiplier process of the n-regular (S, s) economy terminates in finite steps if and only if $na \le 1$; that is,*

$$\omega = 1 \quad \Longleftrightarrow \quad na \le 1. \tag{34}$$

Proof. Since $\phi_{Z_t}(\theta) = \phi_{z_{t-1}}(\phi(\theta)) = \phi^t(\theta) = \phi(\phi_{Z_{t-1}}(\theta))$ from (18) and $\phi_{Z_t}(1) = P(Z_t = 0) = \omega_t$, we have

$$\omega_t = \phi(\omega_{t-1}). \tag{35}$$

For the initial value $\omega_0 = 0$, $\{\omega_t\}_{t=0}^{t}$ satisfies

$$\omega_1 = \phi(\omega_0) = \phi(0) = p_0 \quad \le \quad \phi(1) = 1 \tag{36}$$

$$\omega_2 = \phi(\omega_1) \quad \le \quad \phi(1) = 1 \tag{37}$$

$$\vdots \qquad \vdots \tag{38}$$

$$\omega_t = \phi(\omega_{t-1}) \quad \le \quad \phi(1) = 1. \tag{39}$$

Therefore, $\{w_t\}_{t=0}^{\infty}$ is a bounded monotonic sequence and converges to the limit

$$\lim_{t\to\infty} \omega_t = \omega \le 1 \ . \tag{40}$$

In the limit, ω is a fixed point of ϕ in $[0, 1]$; that is, $\omega = \phi(\omega)$.

We assume that $p_0 > 0$ to avoid a trivial case. Map ϕ is continuous for $z \in [0, 1]$. Further, ϕ is non-decreasing and convex because

$$\phi'(\theta) = \sum_{k=1}^{\infty} k p_k \theta^{k-1} \ge 0 \ , \tag{41}$$

$$\phi''(\theta) = \sum_{k=2}^{\infty} k(k-1) p_k \theta^{k-2} \ge 0 \ . \tag{42}$$

It follows that $\omega = \phi(\omega)$ has either one or two fixed points in $[0,1]$. Obviously, $\omega = 1$ is one of the fixed points,

$$\phi(1) = \sum_{k=0}^{\infty} p_k = 1. \tag{43}$$

In statistical equilibrium, no other fixed point exists in $[0,1]$ if $\phi'(1) = na \leq 1$.

□

Note that this condition for termination of the multiplier process does not satisfy the Brauer–Solow sufficient conditions as well as the Hawkins–Simon conditions in a linear model. We consider a random case where the distribution shows a strictly positive variance. If $na = 1$, the probability is that the multiplier process terminates in finite steps. Nevertheless, the stochastic multiplier diverges in this case.

$$\mathbf{E}(M) = \mathbf{E}\left(\sum_{t=0}^{\infty} Z_t\right) = \sum_{t=0}^{\infty} 1^t = \infty. \tag{44}$$

Technically, if the variance is zero, we would have $\lim_{t\to\infty} P(Z_t = 0) = 0$ in this case. However, this fact indicates that the termination of the multiplier process and convergence of the multiplier are totally different matters. The condition for convergence in the (S, s) economy model could be weaker than that in a linear model.

4.4 Criticality and rate of return

Let q_i be the price of goods i. From the homogeneity of a_{ij}, $q_i = q_j, \forall i, j \in \mathcal{N}$, we define the uniform rate of return, r, as

$$r = \frac{q_j - \sum_{i=1}^{N} q_i a_{ij}}{\sum_{i=1}^{N} q_i a_{ij}} = \frac{1 - na}{na} . \tag{45}$$

Theorem 4.3 (asymptotic divergence). *The stochastic multiplier of the n-regular (S, s) economy is asymptotically divergent when the uniform rate of return is very close, but not equal, to* 0.

Proof. From (45) $na = 1/(1+r)$, we have

$$\mathbf{E}(M) = \frac{1}{1 - \dfrac{1}{(1+r)}} = \frac{1}{r} + 1 . \tag{46}$$

For the limit $r \to 0$,

$$\lim_{r\to 0} \mathbf{E}(M) = \lim_{r\to 0} \left\{ \frac{1}{r} + 1 \right\} = \infty . \tag{47}$$

□

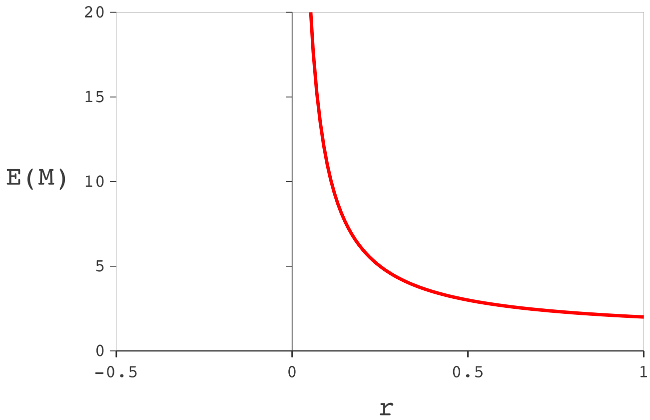

Figure 3: Asymptotic divergence of the n-regular (S, s) economy.

This result tells us that $r = 0$, that is $na = 1$, is the critical point of this economy. This corresponds precisely to Bak et al.'s (1993) case, which says that the system evolves to the state of self-organized criticality wherein the triggered chain reaction is much larger than the original shock and any size of the chain reaction can occur. Their model displays power-law fluctuation and $E(M)$ diverges to infinity. However, in the usual case where $r > 0$, that is, $na < 1$, the economy is said to be subcritical and $E(M)$ is a finite size.

Now, we examine the fluctuation of M. The multiplier process is represented as the branching process of a tree, which is a connected graph with no loops. We take advantage of this nature and calculate the size distribution of the stochastic multiplier.

If $M = m$, it contains m production nodes in the tree constructing the multiplier process, depicted as • in Figure 2. The tree has $m \times n$ branches, and $m - 1$ out of $m \times n$ are triggered to produce while $m \times n - (m - 1)$ are not triggered. With regard to Figure 2, where $n = 2$, the tree has $8 \times 2 = 16$ branches and $m \times n - (m - 1) = 16 - 7 = 9$, the probability of m size of tree can be expressed by $\pi^m(1-\pi)^{m\times n-(m-1)}$, where π is the ratio of inventory level equal to one.

In general, the size distribution of the stochastic multiplier, M, is given by

$$P(M = m) = \frac{1}{m+1}\begin{pmatrix} m \times n \\ m \end{pmatrix} \pi^{m-1}(1-\pi)^{m\times n-(m-1)} \quad \text{with} \quad m = 1, 2, 3, \cdots \tag{48}$$

In order to count the number of trees with m nodes, for the case $n = 2$, the Catalan number C_m is well known,

$$C_m = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} = \frac{(2m)!}{m!\,(m+1)!}\,. \tag{49}$$

For $n = 2$, with this number, we give the size distribution of the stochastic multiplier, M, by

$$P(M = m) = C_m \pi^{m-1}(1-\pi)^{m+1} \quad \text{with} \quad m = 1, 2, 3, \cdots \tag{50}$$

Theorem 4.4 (size distribution). *In statistical equilibrium, the distribution function of multiplier M is analytically given as*

$$P(M = m) \sim \frac{1-a}{a\sqrt{\pi}} \cdot m^{-\frac{3}{2}} \cdot \exp\left(-\frac{m}{\xi(a)}\right) \quad \textit{with} \quad m = 1, 2, 3, \cdots , \tag{51}$$

where the correlation length $\xi(a)$ is

$$\xi(a) \sim \left(\frac{1}{1-2a}\right)^2 . \tag{52}$$

Proof. From Stirling's formula, where m is large enough to replace $m + 1 \sim m$, the Catalan number becomes

$$C_m = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} = \frac{1}{m+1}\frac{(2m)!}{m!(m+1)!} \sim \frac{1}{m+1}\frac{1}{\sqrt{\pi m}}\frac{(2m)^{2m}}{m^{2m}} \tag{53}$$

$$\sim \frac{1}{m+1}\frac{4^m}{\sqrt{\pi m}} \sim \frac{4^m}{\sqrt{\pi}} \cdot m^{-\frac{3}{2}} . \tag{54}$$

In statistical equilibrium, if $\pi = a$, then we have

$$P(M = m) \sim \frac{1-a}{a\sqrt{\pi}} \cdot m^{-\frac{3}{2}} \cdot [4a(1-a)]^m . \tag{55}$$

Because $a < 1/2$, $a(1-a)$ is less than $1/4$, and so $[4a(1-a)]^m$ decreases. Therefore, we have $[4a(1-a)]^m = e^{m \ln[4a(1-a)]}$.

From Theorem 4.3, the critical point of this economy is $a_c = 1/n = 1/2$. Introducing a deviation from the critical point, $D \equiv a - a_c = a - 1/2$, and so we can write

$$a(1-a) = \frac{1}{4} - D^2.$$

We conclude by using the Taylor expansion and plugging the result into (55). □

Figure 4 are graphs of (51), the distribution mass function of multiplier sizes. The (S, s) economy evolves or self-organizes into a statistically stationary state, where the distribution of the multiplier sizes can be given by

$$P(M = m) \sim \begin{cases} m^{-\frac{3}{2}} \cdot \exp\left(-\frac{m}{\xi(a)}\right) & \text{if } a < \frac{1}{2} \\ m^{-\frac{3}{2}} & \text{if } a \to \frac{1}{2} \end{cases} . \tag{56}$$

If m is smaller than $\xi(a)$, the probability mass function is well approximated by power law with exponent $3/2$. However, for a large m, the exponential decay dominates. In the critical case, $a = 1/n$ $(= 1/2)$, the exponential disappears and the distribution is a pure power law.

Parameter a is an input coefficient that is so regular that it can be interpreted as a function of the uniform rate of return r, as seen in Theorem 4.3. If $a < 1/n$ $(= 1/2)$, then $r > 0$ and $(1 - na) > 0$; that is, the profit is positive and the system is subcritical. An external perturbation of demand is absorbed by a certain profit rate. That is, energy is not conserved in this case. Therefore, the correlation length ξ in finite perturbation cannot propagate a long distance.

IF $a = 1/n$ $(= 1/2)$ then $r = 0$, that is, zero-profit and critical point. In this case, external perturbation cannot be absorbed, energy is conserved, and the system is critical since the distribution function (56) shows a power-law distribution.

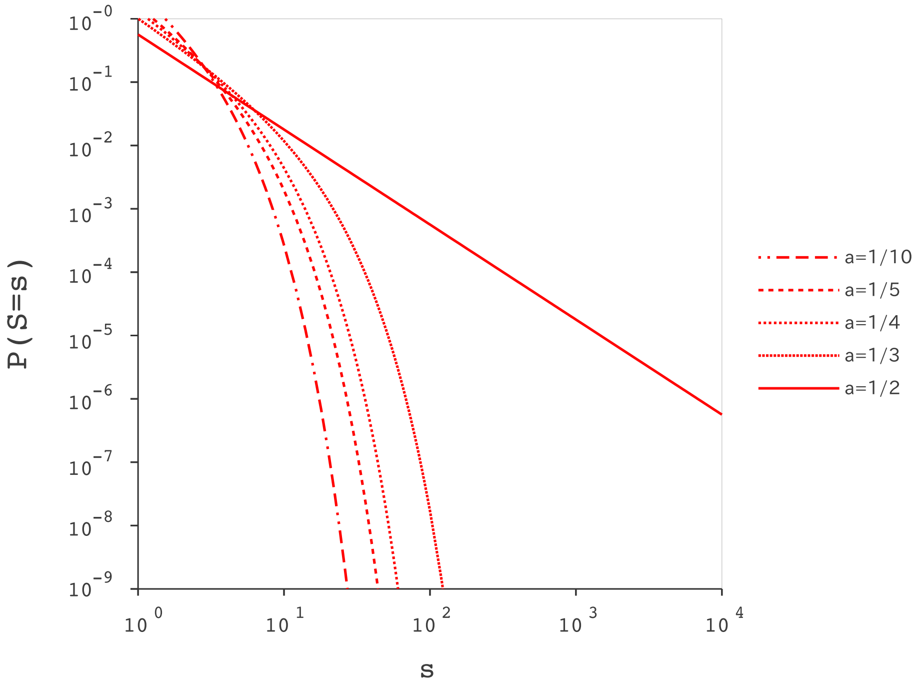

Figure 4: Size probability mass function in n-regular (S, s) economy in the case $n = 2$.

5 Conclusions

In this paper, we proposed a general quantity adjustment behavior and input-output structure model. We employ the (S, s) inventory policy as an inventory adjustment behavior with a non-linear nature. This nature plays an important role in aggregate dynamics. A threshold adjustment allows each firm to store the adjustment energy and productions have lumpy perturbations. Once a production takes place, the firm releases a large amount of energy that propagates through the economy in a multiplier process.

After defining the general framework, we restrict the input-network structure to a regular graph. We define the multiplier process as a branching and derive the expectation value of the multiplier in the statistical steady state. We give key theorems in Section 4; one is the sufficient condition for the convergence of the multiplier in terms of expectations. This condition is similar to the well-known Brauer–Solow condition, $\sum a_{ij} < 1$; it takes the form $na < 1$ in our model. Another is the necessary and sufficient condition for termination of the multiplier process. We prove that the condition is $na \leq 1$. However, for the case of $na = 1$ in particular, this violates Solow's condition as well as the Hawkins–Simon condition. We conclude that the necessary and sufficient condition for termination of the multiplier process is weaker than that in the linear model.

We also prove that the multiplier is finite and that its probability distribution function decays exponentially if the rate of return is positive. Rate of return is associated with the input coefficient a in our model; positive returns mean that $na < 1$. On the other hand, if the rate of return has a zero limit ($na = 1$), the multiplier asymptotically diverges and the probability distribution function is asymptotically a power law. This means that a positive rate of return holds down the high fluctuation of output and moderate business cycles.

References

[1] Arrow, K., Harris, T. and Marschak, J., "Optimal inventory policy," *Econometrica*, 59, 1951.

[2] Bak, P., Tang, C. and Wiesenfeld, K., "Self-organized criticality: an explanation of 1/f noise," *Physical Review Letters*, 59, 1987.

[3] Bak, P., Chan, K., Scheinkman, J. and Woodford, M., "Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics," *Ricerche economiche*, 47, 1993.

[4] Blinder, A S. and Maccini, L J., "Taking stock: a critical assessment of recent research on inventories," *Journal of Economic Perspectives*, 5, 1991.

[5] Caballero, R. and Engel, E., "Dynamic (S,s) economies," *Econometrica*, 59, 1991.

[6] Caplin, A., "The variability of aggregate demand with S,s inventory policies," *Econometrica*, 1985.

[7] Dhar, D., "Self-organized critical state of sandpile automaton models," *Physical Review Letters*, 64,1990.

[8] Harris, T. E. *The theory of branching processes*, Springer-Verlag, 1963.

[9] Keynes, J.M. *The General Theory of Employment, Interest and Money*, Macmillan, 1936

[10] Nirei, M., "(S,s) inventory dynamics and self-organized criticality," *Ricerche economiche*, 54, 2000.

[11] Ramey, V. A. and K. D. West., "Inventories," *Handbook of Macroeconomics*, Vol. 1B, 1999.

[12] Scarf, H., "The optimality of (s,S) policies in dynamic inventory method," *Mathematical Method in the Social Science*, 59, 1960.

[13] Scheinkman, J.A. and Woodford, M., "Self-organized criticality and economic fluctuations," *American Economics Association Papers and Proceedings*, 84, 1994.

[14] Zapperi, S., Lauritsen, K. B. and Stanley, H.E., "Self-organized branching processes: mean-field theory for avalanches," *Phys. Rev. Lett.*, 75, 4071, 1995.